Mean vs. Median-Based Voting in Multi-Dimensional Allocation Problems [*]
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Jana Rollmann

Department of Economics and Management
Karlsruhe Institute of Technology (KIT)
D – 76131 Karlsruhe, Germany
jana.rollmann@kit.edu

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Abstract *We empirically test different voting rules on the allocation of public projects in a laboratory experiment. Specifically, we ask whether voters try to manipulate the outcome to their benefit. The mean rule, according to which the average allocation is chosen, is highly manipulable theoretically, and we find indeed that subjects have a strong tendency to play the corresponding (non-truthful) Nash equilibrium strategy. By contrast, median-based rules are more difficult to manipulate, and sometimes truth-telling is in fact a weakly dominant strategy. The behavior of subjects is more subtle here and, surprisingly, also more difficult to predict. While most subjects play a best response to truth-telling of all other voters, a significant fraction of them does not vote truthfully themselves. We also find that the voting rule has a significant impact on the frequency of Nash-play, truth-telling and the extent of deviation from the truth.*

JEL Classification C72, C91, D71

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1 Introduction

Consider the following situation: There is a football club that drew profit over the previous season and plans to spend it on a number of projects. These projects are public, meaning that everyone in the football team has access to them, like new balls, jerseys or locker rooms. Since it is important to the football club that there are no discrepancies, they want to let everyone in the team take part in the decision on how to allocate the budget. Each team member might reveal his most preferred allocation of the total budget and these votes will be aggregated to result in a social outcome. Emerging questions are for example: How should the votes be aggregated? Given the voting rule, are there incentives not to vote for the true preferred allocation? A feasible rule to aggregate budget allocations is the median rule. Under the assumption of single-peaked preferences[1], Cason et al. (2006) state that "Many [...] strategy-proof mechanisms [...] have Nash equilibrium outcomes that do not coincide with the dominant strategy equilibrium outcome. These Nash equilibrium outcomes are frequently socially undesirable."

Consider three team members that vote on the allocation of 100 monetary units (MU) on two public projects: balls and jerseys. Assume that their most preferred allocations for balls are 25, 40 and 65 MU, respectively.[2] The median rule is a strategy-proof voting rule for aggregating the votes, which implies that truth-telling is a possible Nash equilibrium. Given that every team member reveals his true preferred allocation, the social choice under the median rule is to allocate 40 MU on balls and 60 MU on jerseys, respectively. In this equilibrium, one team member achieves exactly his preferred outcome and therefore given truth-telling of the others, he has no incentive to deviate from revealing his true preferred allocation. The team member who wants to allocate 25 MU on balls would favor a social outcome closer to 25 MU. However, he is only able to increase the social outcome by voting for a higher allocation on balls than the median-value. The same applies to the third team member, who is also not able to influence the social outcome to his favor. Hence, truth-telling is a weakly dominant strategy resulting in a Nash equilibrium. Nevertheless, truth-telling is not the only Nash equilibrium. As long as the non-pivotal team members stay within their rank, i.e. vote for an allocation equal to or higher than the median if the preferred allocation exceeds it and an equal or lower allocation if the true allocation undercuts it, every combination of these votes and truth-telling of the pivotal voter results in a Nash equilibrium with the same outcome as truth-telling of all team members.

But what about the 'bad' Nash equilibria Cason et al. (2006) were talking about? Suppose, all three team members vote for an allocation of no budget on balls and the total budget on jerseys. The social outcome is a median allocation of 0 MU on balls and 100 MU on jerseys, which increases the sum of the absolute distances from 40 to 130.[3] However, no team member has the incentive to deviate from his vote, as the

[1] We assume single-peaked preferences throughout the paper. The property of symmetry and single-peakedness regarding preferences will be explained in detail in section 2.2.

[2] Since a total of 100 MU has to be distributed, this automatically implies that the most preferred allocations for jerseys are 75, 60 and 35, respectively.

[3] The sum of the absolute distances is $|25 - 40| + |40 - 40| + |65 - 40| = 40$ in the first situation and

median remains unchanged given the other votes, denoting that 'bad' Nash equilibria exist when using the median rule.

Yamamura and Kawasaki (2013) are providing a solution to this problem: "If there exist a multitude of bad Nash equilibria for a given [...] median rule, then using [...] average rules can be a comparable alternative [...]." Let's reconsider the example from the football club, this time using the average or mean rule to aggregate the votes. A unique and efficient Nash equilibrium[4] exists, in which at most one team member votes for his true preferred allocation. In this equilibrium, the team member with the lowest preferred allocation on balls votes for the lowest possible value, 0 MU, whereas the one with the highest preferred allocation votes for the highest possible value, 100 MU. In order to achieve a Nash equilibrium outcome that corresponds to the preferred allocation of the 'middle-voter', he has to vote for 20 MU on balls.[5]

So far, there is nothing really new and surprising about the mean or the median rule. But what if the number of public projects increases? When using the median rule, truth-telling is once more a Nash equilibrium. Also, as long as the team member with the pivotal preferred allocation for one project states the truth and the others stay within their rank, every combination of votes results in a Nash equilibrium. Even though the sum of absolute deviations from the true preferred allocation is minimized, the coordinate-wise median may violate the budget restriction. An adaptation of the social outcome is possible, but in general, adapted median rules are not strategy-proof. Besides, 'bad' Nash equilibria, that reduce the utility of every team member compared to the social outcome under truth-telling, do not disappear when using the median rule on more than two public projects. When following the suggestion of Yamamura and Kawasaki (2013) again and making use of the mean rule, the allocation problem on two public projects yields a unique and efficient Nash equilibrium. There exists a focal Nash equilibrium in many cases in multi-dimensional allocation problems as well. What might be surprising is the fact that sometimes a multiplicity of Nash equilibria occurs when using the mean rule, and even inefficient equilibria are possible.

In our theoretical section, we address multi-dimensional budget allocation problems and provide two possible voting aggregation mechanisms: the mean and the median rule. We present a way of adaptation under the median rule to satisfy the budget restriction. Moreover, we examine best strategies and Nash equilibria of the different voting rules. In a laboratory experiment, we analyze the voting behavior in multi-dimensional and repeated budget allocation problems and oppose the voting strategies under the mean and the median rule. In detail, our research questions are whether under the median rule, the dominant strategy is played if it is existent and whether manipulation possibilities are exploited. Under the mean rule, we pose the question whether the Nash equilibrium is played if it is focal. We are also interested in voting behavior and Nash equilibria in more complex situations.

$|25-0|+|40-0|+|65-0|=130$ in the new setting.

[4]Efficiency is defined in detail in section 2.5. We call a social outcome efficient, if it is not smaller (larger) than the smalles (largest) most preferred allocation.

[5]The average allocation results in $\frac{0+20+100}{3}=40$ MU, which is equal to his preferred allocation.

2 Theoretical Model

2.1 Basic Definitions

Consider a set of individuals $I = \{1, ..., n\}$, that decide on the allocation of a budget E on m public projects $J = \{1, ..., m\}$. We claim a budget restriction such that no negative budget might be allocated as well as the total budget has to be spent. The set of feasible allocations is therefore given by $\mathcal{B} := \{x \in \mathbb{R}^m_{\geq 0} | \sum_{j \in J} x_j = E\}$. The preferences of each individual are metrically single-peaked, as described in detail in section 2.2. A most preferred allocation of individual i is called his peak $p^i = (p^i_1, \dots, p^i_m) = (p^i_j)_{j \in J} \in \mathcal{B}$, the vector of all peaks is given by $p = (p^1, \dots, p^n) = (p^i)_{i \in I}$. In order to decide on the social outcome of the allocation, every individual submits a vote $q^i = (q^i_1, \dots, q^i_m) = (q^i_j)_{j \in J} \in \mathcal{B}$ that might differ from his peak. The vector of all votes is $q = (q^1, \dots, q^n) = (q^i)_{i \in I}$. We measure the distance between individual i's peak and the social outcome $x(q) = (x_1(q), \dots, x_m(q)) \in \mathcal{B}$ with the L_1 distance, i.e. the sum of their absolute deviation in every project: $d(p^i, x(q)) := \sum_{j \in J} |p^i_j - x_j(q)|$. We exclude negative or budget-exceeding peaks, votes and social outcomes and make sure that the budget is satisfied as we claim $p^i, q^i, x(q) \in \mathcal{B}$.

Let the number of public projects be $m = 3$. The budget constraint is graphically given by a triangle in a 3-dimensional space, see figure 1a. This simplex contains all possible allocations, at which the budget constraint is satisfied, i.e. all allocations in $\mathcal{B}$. The vertices are the allocations at which the total budget is distributed on exactly one project and no budget on the other two projects. On the lines between the vertices, a budget of zero is allocated on one project. As can be seen in figure 1b, it is possible to represent the 3-dimensional simplex as a 2-dimensional one. By doing so, it is obvious that a shift from one point to another inside the simplex means re-allocating the budget, such that the budget constraint is not violated.

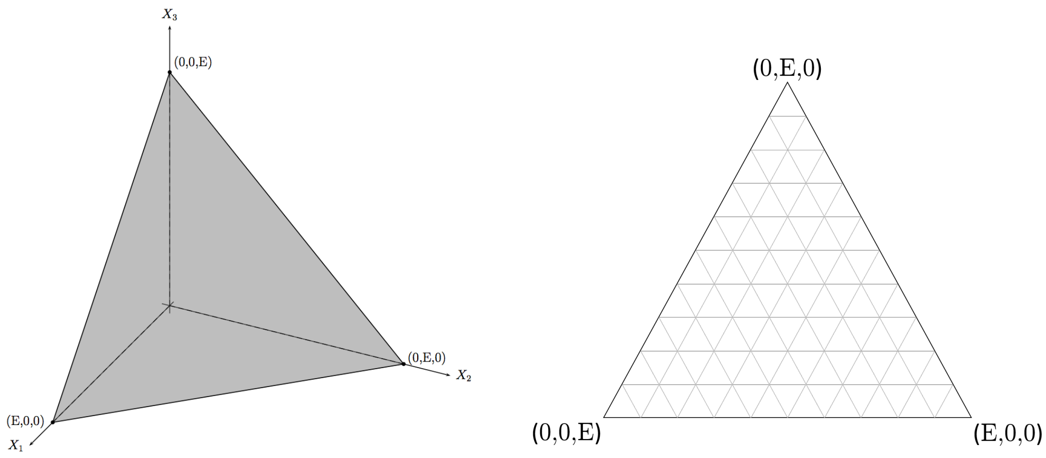
2.2 The Voters' Preferences

In order to determine the best strategy for an individual, his preferences need to be specified. We use the L_1 distance function that sums up the project-wise absolute differences between two allocations $a, b \in \mathcal{B}$. This sum is defined as distance $d(a, b)$ between the two allocations a and b. Every voter i is assumed to have a preference ranking that satisfies the following:

- there exists a unique peak p^i and
- voter i (weakly) prefers a over b if and only if $d(p^i, a) \leq d(p^i, b)$ for all $a, b \in \mathcal{B}$.

Preference rankings that satisfy the above conditions are single-peaked in every possible direction.[6] This implies that voters try to minimize the distance between the social outcome and the true peak, $d(p^i, x(q))$ meaning that a lower distance from the peak

[6] Nehring et al. (2008) define these preferences as metrically single-peaked, see also Lindner (2011).



(a) 3-dimensional space (b) 2-dimensional space

Figure 1: The simplex

results in a higher payoff and the payoff reaches its maximum when $p^i = x(q)$. Moreover, we make sure that the budget allocation of every project is equally important. Thus, a deviation of $x(q)$ from p^i in project 1 has the same impact on the payoff as an equally high deviation from p^i in project 2 or 3. Figure 2 shows the indifference curves of individual i given his peak p^i. The highest payoff is given for a social outcome equal to p^i and it decreases symmetrically in every project with a higher distance from the peak, which is represented by the hexagon-shaped curves.

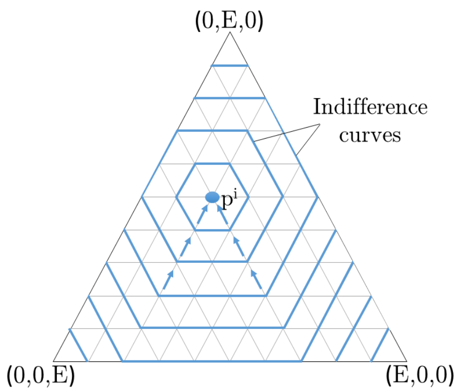


Figure 2: Single-peaked preferences

2.3 The Mean Rule

The social outcome under the mean rule is calculated by adding up the votes of all individuals separately for every project and dividing these sums by the number of votes:

$$Mean_j(q) = \frac{1}{n} \sum_{i=1}^{n} q_j^i. \tag{1}$$

Example 2.1. $E = 100;\ q^1 = (20, 50, 30);\ q^2 = (10, 40, 50);\ q^3 = (0, 0, 100)$
$\rightarrow Mean(q) = (10, 30, 60)$

By construction, the social outcome under the arithmetic mean always satisfies the budget constraint, i.e. the sum of the coordinate by coordinate mean-values adds up to the allocatable budget. Another property of the mean rule is its manipulability. Given different peaks, at most one individual votes for the true preferred allocation in a Nash equilibrium, as shown in section 2.6.1.

2.4 The Median Rule

The median rule selects of all ordered votes $q_j^{[i]}$ for every project the one in the middle if the number of individuals is odd or the average of the two middle votes when there is an even number of voters. Thus, the median consists of m coordinate by coordinate median-values:

$$Med_j(q) = \begin{cases} q_j^{[\frac{n+1}{2}]}, & \text{if } n \text{ is odd} \\ \frac{1}{2} \cdot (q_j^{[\frac{n}{2}]} + q_j^{[\frac{n}{2}+1]}), & \text{if } n \text{ is even.} \end{cases} \tag{2}$$

Example 2.2. $E = 100;\ q^1 = (70, 30, 0);\ q^2 = (10, 40, 50);\ q^3 = (20, 60, 20)$
$\rightarrow Med(q) = (20, 40, 20)$

A restriction to the median rule is the possibility that the coordinate by coordinate median-values do not satisfy the total budget in multi-dimensional allocation problems, i.e. $\sum_{j=1}^{m} Med_j(q) \neq E$ for $m > 2$. In the previous example 2.2 the total budget is undercut and therefore an adaptation of the median outcome is necessary.

The Normalized Median Rule

The normalized median rule, suggested by Nehring et al. (2008), chooses the element on the simplex, at which the values of the coordinates are in the same proportion to each other compared to the original median-values. Graphically, the normalized median is the allocation that lies on the simplex as well as on the conduit through the zero point and the median, as shown in figure 3. Computationally, the normalized median per coordinate is determined by a multiplication of the corresponding median-value with the total budget divided by the sum of the median-values for all coordinates before adaptation:

$$NMed_j(q) := \begin{cases} Med_j(q) \cdot \frac{E}{\sum\limits_{j \in J} Med_j(q)}, & \text{if } Med_j(q) > 0 \text{ for at least one } j \\ \frac{E}{m}, & \text{else.} \end{cases} \tag{3}$$

Example 2.3. $E = 100;\ Med(q) = (20, 40, 20)$
$\rightarrow NMed(q) = (20, 40, 20) \cdot \frac{100}{80} = (25, 50, 25)$

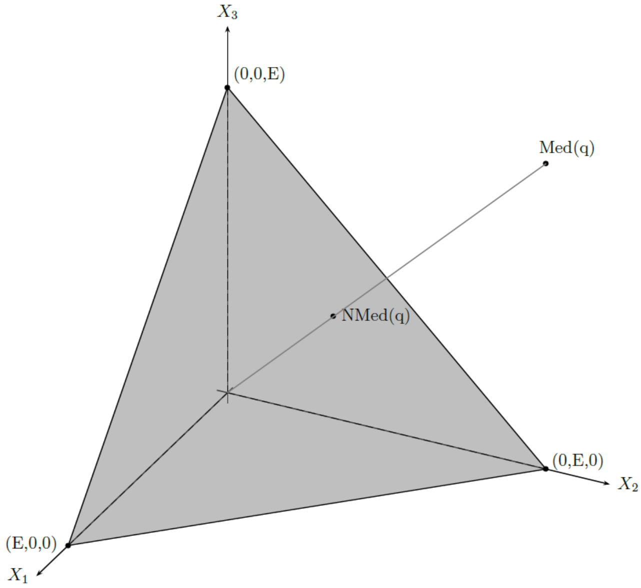


Figure 3: The normalized median

2.5 Efficiency of Social Outcomes

As already mentioned, we evaluate the voting rules by their efficiency. Therefore, we need a more detailed definition and classification of how efficient the social outcome is.

Definition 2.1. *A social outcome $x(q)$ is called **efficient**, if $x_j(q)$ is not smaller than the lowest ranked peak and not larger than the highest ranked peak for each project j, i.e. $p_j^{[1]} \leq x_j(q) \leq p_j^{[n]}$ for all j.*

When considering a budget allocation problem on two public projects, a social outcome is efficient, if it lies within the convex hull of all peaks. In multi-dimensional budget allocation problems, the convex hull comprises indeed also only efficient outcomes, however, the set of efficient outcomes is even larger. Figure 4 displays the difference between the convex hull and the set of efficient outcomes for $m = 3$. Moreover, when voting on a budget allocation on more than two public projects, the set of efficient outcomes as well as the convex hull of the peaks might include outcomes that are Pareto-inefficient.

Definition 2.2. *A social outcome $x(q)$ is called **Pareto-efficient**, if there exists no other social outcome $x'(q)$, such that the distance between $x'(q)$ and p^i is shorter compared to the distance between $x(q)$ and p^i for at least one individual i and not greater for all other individuals: $d(p^i, x(q)) < d(p^i, x'(q))$ for at least one i and $d(p^i, x(q)) \leq d(p^i, x'(q))$ for all other i.*

Therefore, an outcome satisfies Pareto-efficiency, if it is equal to the median of all peaks for at least two coordinates. Deviating from this outcome always puts at least one voter in a worse position. A Pareto-efficient outcome must not necessarily be the 'best' social outcome for the group in total. There exists a smaller set of outcomes, that minimizes the total distance sum over all individuals and thus represents a welfare optimum, as can be seen in figure 4.

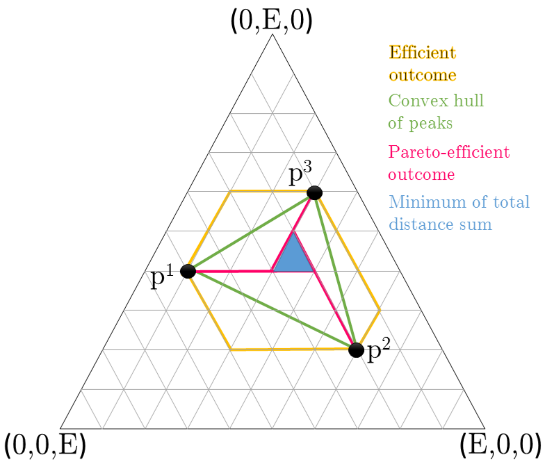


Figure 4: Efficiency

2.6 Individual Strategies

With the knowledge on the voting rules, one should think of possible strategies during the voting process. A simple and straightforward strategy is truth-telling, i.e. stating a vote q^i that corresponds to the peak p^i. Nevertheless, for some individuals there are incentives to deviate from the truth if the social outcome might be influenced to their benefit. Given the votes of the other subjects, individual i can influence the social outcome within his option set. Beside, the option set depends on the 'weight' of individual i and therefore on the number of total votes n:

$$\mathcal{OS}^i(q^{-i}) := \left\{ x \in \mathcal{B} \mid \exists\ q^i \in \mathcal{B} : x = \frac{1}{n} \cdot \left(\sum_{k \in I \setminus \{i\}} q^k + q^i \right) \right\} \tag{4}$$

The option set can be displayed as a triangular in the simplex, see figure 5.

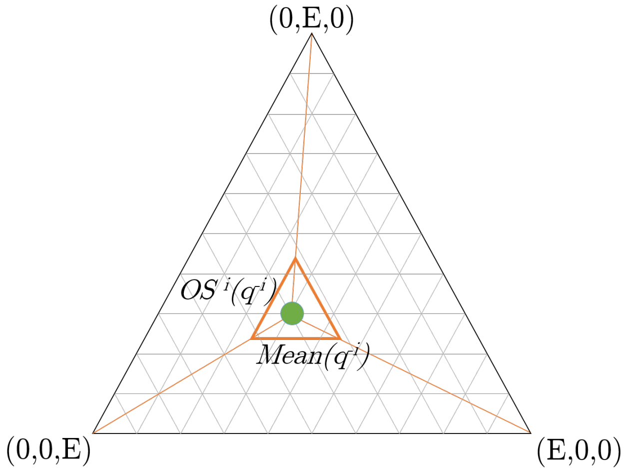


Figure 5: The option set of individual i

Definition 2.3. *A vote q^i of one individual i that differs from the peak p^i and decreases the distance between p^i and the social outcome $x(q)$ is called **manipulation**.*

For the analysis of best strategies, we suppose perfect information on the true peaks of the other voters. Best strategies may not be unique but might comprise ranges of allocations. Due to the construction of the voters' preferences, given the votes of the others, there might be several best responses of an individual that result in the same distance and therefore yield equal utilities. Moreover, we assume that there are no manipulation costs, which among others implies that lying itself does not affect the payoff (whereas a shift in the social outcome due to a changed vote might). However, one might argue that lying decreases the payoff not directly but in a more subtle and moral way that is unobservable or unquantifiable. We disregard these kind of costs.

2.6.1 Nash Equilibria of the Mean Rule

Under the mean rule, manipulation is possible for most peak distributions.[7] For the following analysis, we assume $m = 3$. Suppose there exists a total budget of $E = 100$ and $i = 2$ individuals with the peaks $p^1 = (35, 60, 5)$ and $p^2 = (22, 15, 63)$. Given both state their true preferred allocation, the social outcome $x(q)$ is $Mean(q) = (28.5, 37.5, 34)$, resulting in distances from the peaks of each individual of $d(p^i, x(q)) = 58$. If the first individual reallocates 5 units from the third to the second project, the mean outcome changes to $Mean(q) = (28.5, 40, 31.5)$, decreasing the distance from the first individual's peak to 53 (and increasing the distance from the second one's to 63). Analogically, the second individual has an incentive to deviate from truth-telling and is able to manipulate by shifting units from the second to the third project. We can not only state that manipulation is possible, but also in which direction and to what extent.

[7]Exceptions include equal peaks of all voters and peaks that allocate zero budget to at least one project.

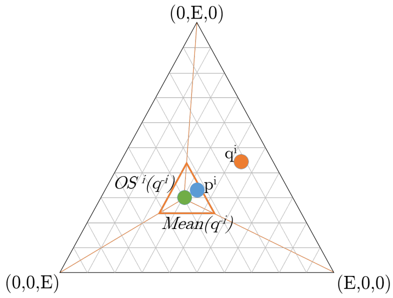
Given the votes of the other participants, individual i might face two different scenarios. In the first one, $p^i \in \mathcal{OS}^i(q^{-i})$, such that i is able to manipulate the social outcome in a way that it corresponds to his peak. Therefore, the triangular-shaped option set is dilated by the factor n on the size of the simplex. Given the mean outcome of the other votes, individual i's best response is to vote for an allocation, such that $Mean(q) = p^i$. In this situation, q^i is strictly positive for every project as long as p^i lies on the inside of the option set (see figure 6a), zero for one project, if p^i lies on an edge of the option set (see figure 6b) and zero for two projects, if p^i lies on a vertex of the option set (see figure 6c). In summary, if $p^i \in \mathcal{OS}^i(q^{-i})$, then $p^i = Mean(q)$ can be achieved by voting either $q^i_j > 0$ for all j, $q^i_j = 0$ for one j or $q^i_j = 0$ for two j, depending on the location of the peak in the option set.

In the second scenario, $p^i \notin \mathcal{OS}^i(q^{-i})$, as displayed in figure 6d. Individual i might now only manipulate to the extent that he minimizes the distance between the social outcome and his peak. In order to achieve the lowest possible distance, he submits a vote q^i, which results in a mean outcome that is tangent to his closest hexagon-shaped indifference curve. The best response is unique, if the closest tangent is a vertex of the option set. In this case, $q^i_j = 0$ for two j. If the intersection between indifference curve and option set is a line segment, the optimal choice of individual i is not unique, since he might vote for a set of allocations that result in social outcomes with the same distance to his peak. Here, a best response is voting zero for at least one project.

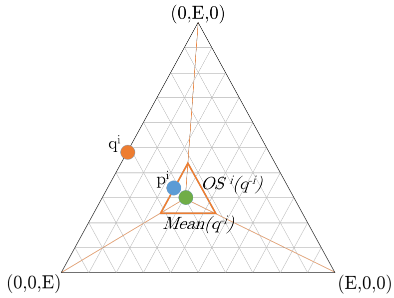
Observation 1. *Given a budget allocation problem on $m = 3$ public projects. In every Nash equilibrium of the mean rule, if all individuals have different peaks, at most one individual votes for a strictly positive amount of every public project, i.e. for at most one voter i, $q^i_j > 0$ for all j.*

Consider a situation in which *two* individuals vote for a strictly positive amount of every public project. Since we assume different peaks, at most one individual's peak is equal to the social outcome. If the social outcome differs from the peak, the mean-value deviates from the preferred allocation in at least two projects. Moreover, in at least one project, the mean-value is larger than the own preferred allocation, since the total budget has to be allocated by the peak as well as by the social outcome (i.e. $p^i, Mean(q) \in \mathcal{B}$). The best strategy in this situation is to shift the allocation in order to vote for a smaller amount for these projects. The other individual, that also voted for a strictly positive amount of every project, might manipulate the outcome and will behave accordingly if his peak differs from the social outcome. As long as the outcome is not equal to one of the peaks, both individuals have an incentive to allocate a smaller amount to the project for that $p^i_j < Mean_j(q)$ holds, until the smallest possible vote for this project is reached, namely zero. Therefore, in a Nash equilibrium, there is maximal one individual, who does not vote zero for at least one project: the individual, that can achieve a social outcome equal to his preferred allocation. Thus, given different peaks of all individuals, at most one voter announces his true peak p^i in a Nash equilibrium and at least $(n-1)$ individuals vote zero for at least one project.

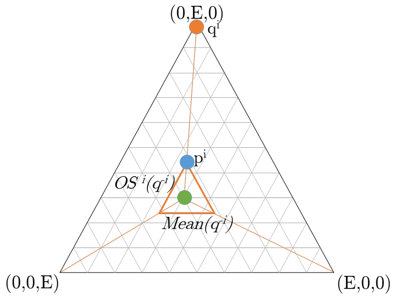
Block (2014) as well as Bauer and Puppe (2016) show that under the mean rule, there exists a unique Nash equilibrium, which is efficient. Nevertheless, this statement is only



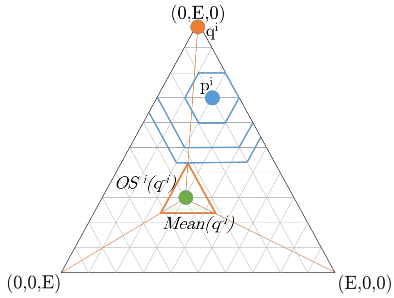
(a) Peak within option set



(b) Peak on edge of option set



(c) Peak on vertex of option set



(d) Peak outside of option set

Figure 6: Option sets

true for one-dimensional allocation problems. With more than two public projects, the uniqueness and efficiency of the Nash equilibrium might dissolve, as there are multiple Nash outcomes including some that increase the distance between the social outcome and all peaks.

Observation 2. *Given a budget allocation problem on $m = 3$ public projects. Under the mean rule there might exist several Nash equilibria, including Nash outcomes that lower the utilities of all voters.*

Assume that under the mean rule and $m = 3$ public projects there exists only one Nash equilibrium that is always unique and efficient. The following example provides a contradiction.

Example 2.4. *Consider a budget of $E = 100$ that has to be allocated on $m = 3$ public projects using the mean rule. The voters' peaks are as follows: $p^1 = (30, 10, 60)$, $p^2 = (10, 30, 60)$ and $p^3 = (20, 20, 60)$. A Nash equilibrium consists of the votes $q^1 = (10, 0, 90)$, $q^2 = (0, 10, 90)$ and $q^3 = (50, 50, 0)$, with a social outcome equal to the preferred allocation of the third voter, $Mean(q) = (20, 20, 60)$. In this situation, no voter can improve himself by deviating from his vote and the sum over all individual distances between social outcome and peak is 40, as $d(p, x(q) = (20; 20; 0)$. However, there exists another Nash equilibrium that is inefficient: $q^1 = (90, 0, 10)$, $q^2 = (0, 90, 10)$ and*

$q^3 = (0, 0, 100)$, *leading to a social outcome of* $Mean(q) = (30, 30, 40)$. *The absolute deviation from the peaks is higher for every voter,* $d = (p, x(q) = (40; 40; 40))$, *with a total sum of* 120.

In games with multiple Nash equilibria, Schelling (1960) introduces the concept of focal points. Focal points are solutions with outstanding character, such that players expect of others to play the prominent strategy. Following Myerson and Weber (1993), we call a Nash equilibrium focal, if all players act according to these expectations. In terms of our concrete budget allocation problem, we say that an equilibrium strategy is focal if it is to vote for the highest possible amount in exactly the project that comes with the highest peak value.

2.6.2 Nash Equilibria of the Median Rule

In order to determine individual strategies and Nash equilibria of the median rule, a definition of pivotality is necessary.

Definition 2.4. *Given a profile* $q \in \mathcal{B}^n$, *voter* $i \in I$ *is* ***pivotal*** *with respect to project* $j \in J$ *at* q *if for all* $\varepsilon > 0$ *there exists* $\hat{q}^i \in U_\varepsilon(q^i)$ *such that* $x_j(q) \neq x_j(\hat{q}^i, q^{-i})$, *with* $U_\varepsilon(q^i) := \{x \in \mathcal{B} : d(x; q^i) < \varepsilon\}$.

Example 2.5. $E = 100$; $m = 3$; $p^1 = (20, 50, 30)$; $p^2 = (20, 20, 60)$; $p^3 = (10, 20, 70)$. *Given the median rule and* $q^1 = p^1$, $q^3 = p^3$, *voter two is pivotal with respect to the third project, such that* $p_3^2 \pm \varepsilon$ *will change the social outcome. Voter two might as well change the social outcome of the first and the second project by* $p_1^2 - \varepsilon$ *and* $p_2^2 + \varepsilon$. *Analogously, voter one is also pivotal with respect to the first and voter three with respect to the second project.*

Observation 3. *Given a budget allocation problem on* $m = 3$ *public projects. Truth-telling is a Nash equilibrium of the normalized median rule if there exists one voter who is pivotal with respect to all projects* $j \in J$.

Consider a situation in which one voter is pivotal with respect to every public project. Given truth-telling of every individual, the median outcome is equal to the peak of this pivotal voter. Since the peaks satisfy the budget, in this case the same is true for the median outcome and normalization is not necessary. The social outcome is equal to the peak of the pivotal voter, leading to the lowest possible distance, such that revealing the true preferred allocation is the best strategy. For the non-pivotal voters or voters, that are pivotal with respect to $m - 1$ projects, beneficial manipulation is not possible. The only way to change the social outcome is to vote for an allocation that increases the distance from the own peak, such that these individuals do not have an incentive to deviate from truth-telling. Given the impossibility for every voter to improve his utility by voting for a different allocation than his peak, truth-telling is a Nash equilibrium if one voter is pivotal with respect to all projects.

Observation 4. *Given a budget allocation problem on* $m = 3$ *public projects. If there exists no voter who is pivotal with respect to all projects, manipulation might be possible under the normalized median rule.*

Assume that beneficial manipulation is not possible. The following example provides a contradiction.

Example 2.6. $E = 100;\ m = 3;\ p^1 = (100, 0, 0);\ p^2 = (0, 100, 0);\ p^3 = (1, 1, 98).$ *Individuals one and two are pivotal with respect to the third project, whereas individual three is pivotal with respect to the first and second project. Truth-telling of all individuals leads to a social outcome of* $Med(q) = (1, 1, 0)$*, or* $NMed(q) = (50, 50, 0)$ *after normalization. The distances of the three individuals are* $d(p, x(q)) = (100; 100; 196)$*. However, truth-telling is not a Nash equilibrium. Given truth-telling of the others, individual three is able to manipulate the social outcome. Since he is pivotal with respect to the first and second project, voting for* $q^3 = (0, 0, 100)$*, leads to a social outcome of* $Med(q) = (0, 0, 0)$*, or* $NMed(q) = (33.\bar{3}, 33.\bar{3}, 33.\bar{3})$*. For the other individuals, beneficial manipulation is not possible, resulting in a Nash equilibrium with distances* $d(p, x(q)) = (133.\bar{3}; 133.\bar{3}; 129.\bar{3})$*.*

3 Experiment on the Mean and Median Rule

In a laboratory experiment, we analyze the voting behavior in multi-dimensional and repeated budget allocation problems and oppose the voting strategies under the mean and the median rule.

3.1 Hypotheses

The experiment seeks to test the following hypotheses, which we assemble into three groups:

The Mean Rule

H1.1 Under the mean rule, the Nash equilibrium will be played.

H1.2 Nash-play increases over time under the mean rule.

H1.3 Full information reduces truth-telling under the mean rule.

The Median Rule

H2.1 Under the normalized median rule, truth-telling prevails.

H2.2 Under the normalized median rule, best-response-to-truth is played.

Mean versus Median Rule

H3.1 The deviation of votes from the true peak is higher under the mean than under the normalized median rule.

H3.2 The normalized median rule leads to more truth-telling than the mean rule.

3.2 Experimental Setup, Laboratory Procedure and Design

The experiment took place at the KD2Lab of the Karlsruhe Institute of Technology (KIT) in October 2015. In order to test the hypotheses, eight sessions are conducted that last about 1.25 hours each. The recruitment of the participants is made by ORSEE (see Greiner (2004)). As our experiment is arranged in groups of five, three cohorts, i.e. 15 subjects (120 in total) participate. The average age of the participants is 24.1, the share of women 24.2%. 52.5% study Business Engineering and more than 70% have an economic part in their field of study. These shares correspond to the ORSEE subject pool of the KIT.

Directly after arrival, the participants are allocated randomly to the cabins. The workplaces are equipped with computer, paper, pencil and calculator. The software used for the experiment is zTree (see Fischbacher (2007)). As soon as all 15 participants sit at their workplaces, the instruction that is given to them at registration is read out loud by audiotape to avoid variation in the readings across sessions. Beside some mathematical information on the calculation of the mean or normalized median rule, it includes the session procedure. Subsequently, participants answer a short quiz to make sure the task is understood properly. The instruction can be reread on the handout during the entire study. After the experiment, participants are asked to fill out a questionnaire on demographic data and the strategy underlying their decisions. The average payoff of the participants amounts to 13.98 Euros, including a show-up fee of 5.00 Euros.

The subjects are told that they attend an election together with four other anonymous participants at which the funding of three public projects is determined. Therefore, the participants vote on the allocation of 100 monetary units on the three projects. Since the experiment is designed as between-subjects, the election is either done by mean (the first four sessions) or normalized median rule (the second four sessions). Every participant receives an individual peak that represents the preferred funding of the projects. Thereby, we make sure that the 'true' peaks are known to us. We vary the stated information from *no info*, where the participants obtain only their own peak, to *full info*, where the peaks of the other four participants are displayed. These peak distributions, i.e. five different peaks that belong to the voters of one group, remain the same for several rounds: we repeat the peak distributions of the no info five and the full info treatment three periods. The peak distributions, degree of information and number of periods are identical for both voting rules.

The underlying individual payoff function in the unit ECU[8] is the following:

$$f^i(p^i, x(q)) = 10 + \frac{760}{4 + \sum_{j=1}^{3} |p^i_j - x_j(q)|} \ , \tag{5}$$

where p^i_j denotes the peak of individual i for one of the three projects j and $x_j(q)_j$ the social choice, calculated either by the mean or normalized median of all five votes. One vote represents the share of 100 monetary units that should be allocated to three projects and therefore consists of three natural numbers that have to add up to 100.

[8]Experimental Currency Unit; 100 ECU corresponds to 1.00 Euro.

The minimum payoff per period is 13.73 ECU, because the highest possible distance is 200, e.g. between a social choice $x(q) = (100, 0, 0)$ and a peak $p^i = (0, 100, 0)$. Once the social outcome corresponds to the given peak, the maximal payoff of 200 ECU is reached. Figure 7 shows the payoff function, which is also displayed during the experiment. 'Distance' adds up the absolute distance between the own peak and the social choice in every project, i.e. $\sum_{j=1}^{3} |p^i_j - x_j(q)_j|$.

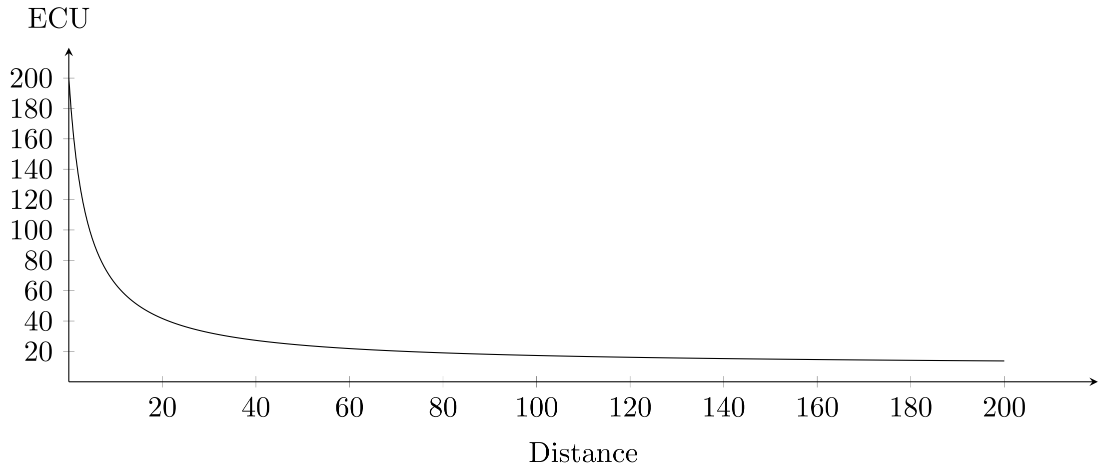


Figure 7: The payoff function

The election is done anonymously and in several rounds. In a first step, the participants get to know their peaks and under full information the peaks of the four other participants of their group. As soon as all participants submit their votes, the subjects get to know the social outcome and their payoff. In total, four peak distributions are used of which three different ones are played in every session. The detailed peak distributions by session are displayed in table 1, where the voting rules of sessions one to four is the mean and of sessions five to eight the normalized median rule. The peak distributions remain the same for five periods in the no info treatment and for three periods in the full info treatment. Accordingly, each participant has to make 24 decisions, consisting of three natural numbers that add up to 100.

Table 1: Peak distributions

Session	Information	Peak Distribution	Peaks by Participant p^1	p^2	p^3	p^4	p^5	Number of Periods
1;5	No Info	A	10	5	70	20	10	5
			65	10	10	20	8	
			25	85	20	60	82	
		B	75	70	20	12	40	
			10	15	15	78	30	
			15	15	65	10	30	
		C	20	15	20	30	10	
			10	30	20	15	20	
			70	55	60	55	70	
	Full Info	A	20	10	10	70	5	3
			20	8	65	10	10	
			60	82	25	20	85	
		B	12	40	70	75	20	
			78	30	15	10	15	
			10	30	15	15	65	
		C	20	20	30	10	15	
			20	10	15	20	30	
			60	70	55	70	55	
2;6	No Info	A	5	10	20	10	70	5
			10	8	20	65	10	
			85	82	60	25	20	
		B	40	20	12	75	70	
			30	15	78	10	15	
			30	65	10	15	15	
		D	50	15	25	25	10	
			30	60	70	50	20	
			20	25	5	25	70	
	Full Info	A	10	20	5	70	10	3
			8	20	10	10	65	
			82	60	85	20	25	
		B	12	75	20	70	40	
			78	10	15	15	30	
			10	15	65	15	30	
		D	25	15	50	10	25	
			70	60	30	20	50	
			5	25	20	70	25	

3;7	No Info	D	15	25	25	50	10	5
			60	70	50	30	20	
			25	5	25	20	70	
		B	75	70	20	12	40	
			10	15	15	78	30	
			15	15	65	10	30	
		C	20	30	20	15	10	
			20	15	10	30	20	
			60	55	70	55	70	
	Full Info	D	25	15	10	25	50	3
			70	60	20	50	30	
			5	25	70	25	20	
		B	70	40	12	75	20	
			15	30	78	10	15	
			15	30	10	15	65	
		C	10	20	30	20	15	
			20	10	15	20	30	
			70	70	55	60	55	
4;8	No Info	A	5	70	10	20	10	5
			10	10	65	20	8	
			85	20	25	60	82	
		D	15	10	25	50	25	
			60	20	50	30	70	
			25	70	25	20	5	
		C	20	30	20	10	15	
			20	15	10	20	30	
			60	55	70	70	55	
	Full Info	A	10	20	70	10	5	3
			8	20	10	65	10	
			82	60	20	25	85	
		D	50	25	10	25	15	
			30	50	20	70	60	
			20	25	70	5	25	
		C	10	20	30	15	20	
			20	10	15	30	20	
			70	70	55	55	60	

3.3 Results

3.3.1 The Mean Rule

Truth-telling

When using the mean rule, we find that participants rarely reveal their true peak. Only 5.7% of all votes are equal to the peaks and truth-telling is low over all four peak distributions.[9] Moreover, truth-telling decreases over time, especially without any information

[9] Note that by truth-telling we only consider peaks of inidividuals for which choosing the true allocation is no Nash-play.

on the peaks of the other participants. While 21.1% of the votes are equal to the individual peaks in the first period, this number declines to 2.3% in the fifth period. We are also able to reject the hypothesis that the share of truth-telling without any information is the same compared to full information, in support of the alternative hypothesis that the mean share of truth-telling is significantly higher in the no info treatment, which supports hypothesis H1.3.

In order to get a better insight of the influencing factors on truth-telling, we run a regression of the 'Peak-Vote-Distance', i.e. the absolute deviation of the vote from the true peak, on a variety of independent variables. The detailed results can be found in table 2. As stated before, we find a positive and significant correlation with the variable 'period', indicating a higher degree of deviation from truth-telling over time. Since the coefficient of 'round', which labels the total decisions over all periods from 1 to 24, is positive and significant, there is not only a higher degree of lying over periods but also over the entire decision-making process. As anticipated, truth-telling decreases slightly with an increasing distance between the true peak and the theoretical Nash-play, since a greater 'Peak-Nash-Distance' indicates that participants have to deviate more from their true peak to play their Nash-strategy.

Contrary to our expectation, the deviation from the peak is positively affected by the 'Nash-Vote-Distance', i.e. the distance between theoretical Nash-play and actual vote. This implies that the higher the deviation of the vote from Nash-play, the more participants tend to lie. In other words, the degree of lying is higher if voters don't behave according to the predicted Nash-strategy. Given Nash-play of the other four voters, this strategy results in a lower payoff. After the experiment, we asked the participants about their approach of voting. Some argued that they tried to deceive the others by votes that lead to a lower payoff in order to receive a higher payoff in the next period. This behavior might explain the results that seem non-strategic at first appearance. Participants also tend to significantly lower truth-telling with a higher distance between the peak and the result of the previous round ('PPR_Dist'). Thereby, we are able to observe a learning effect over periods of increasing manipulation. Although in absolute numbers the difference of truth-telling across the peak distributions is low, we find a significant and high difference in the extent of truth-telling dependent on the theoretical Nash-play. The dummy variable 'edgetruth_d' takes the value 1 if the peak distribution contains an individual Nash-play of either truth-telling (peak distribution D) or voting zero for exactly one project (peak distribution B) and the value 0 if Nash-play of all subjects is to vote zero for two projects. We find that the deviation from the true peak is 26.2 times lower (24.4 times when including answers on the questionnaire to the regression) if the theoretical Nash-play is truth-telling or voting zero for exactly one project, compared to voting zero for two projects. We conclude that truth-telling is higher if the theoretical Nash-play is not focal, like a vote that allocates the total budget in the project with the highest share of the true peak.

Table 2: Regression results mean: peak-vote-distance

VARIABLES	(1) PeakVoteDist	(2) PeakVoteDist
per	3.091***	3.114***
	(0.849)	(0.842)
subj	-0.0633	-0.131
	(0.193)	(0.207)
round	0.697**	0.617*
	(0.331)	(0.333)
PeakNashDist	0.171***	0.161***
	(0.0322)	(0.0321)
NashVoteDist	0.181***	0.190***
	(0.0154)	(0.0155)
PPR_Dist	0.0279**	0.0304***
	(0.0115)	(0.0114)
edgetruth_d	-9.873***	-9.385***
	(3.024)	(3.006)
peak_d	-26.18***	-24.96***
	(2.957)	(2.992)
info_d	5.862	7.049
	(4.668)	(4.678)
male_d		3.309
		(2.178)
understandrule_d		3.291
		(2.104)
econstudy_d		7.996***
		(1.905)
Constant	17.69***	7.884*
	(3.577)	(4.053)
Observations	1,440	1,440
R-squared	0.183	0.200

Standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Nash-play
Hypotheses H1.1 and H1.2 state that we expect Nash-play under the mean rule and that Nash-play increases over time. Table 3 represents the theoretical Nash equilibria of the peak distributions we used in the experiment.

We observe indeed a high ratio of votes (35.8%) that correspond to the theoretical Nash equilibrium. When adding a tendency to Nash-play, which comprises votes with a sum of absolute distance to Nash-play of maximal ten ($\sum_{j\in J} d(\textit{Nash-play}^i_j, q^i_j) \leq 10$), 48.5% of all votes are Nash-play and Nash-tendency. Further, we perceive a learning effect over periods, both with and without information on the other peaks, indicating a convergence to the group Nash equilibrium. Nevertheless, the group Nash equilibrium,

Table 3: Nash equilibria

	Individual				
A	**1**	**2**	**3**	**4**	**5**
p^i	20	10	10	70	5
	20	65	8	10	10
	60	25	82	20	85
q^i_{Mean}	0	0	0	100	0
	0	100	0	0	0
	100	0	100	0	100
q^i_{NMed}	≥10	=10	=10	≥10	≤10
	≥10	≥10	≤10	=10	=10
	=30	≤30	≥30	≤30	≥30
B					
p^i	40	75	70	20	12
	30	10	15	15	78
	30	15	15	65	10
q^i_{Mean}	0	100	100	0	0
	50	0	0	0	100
	50	0	0	100	0
q^i_{NMed}	=20	≥20	≥20	≤20	≤20
	≥15	≤15	= 15	=15	≥15
	≥15	=15	=15	≥15	≤15
C					
p^i	10	20	20	30	15
	20	10	20	15	30
	70	70	60	55	55
q^i_{Mean}	0	0	0	100	0
	0	0	0	0	100
	100	100	100	0	0
q^i_{NMed}	≤20	=20	=20	≥20	≤20
	=20	≤20	=20	≤20	≥20
	≥60	≥60	=60	≤60	≤60
D					
p^i	10	50	25	15	25
	20	30	50	60	70
	70	20	25	25	5
q^i_{Mean}	0	100	25	0	0
	0	0	50	100	100
	100	0	25	0	0
q^i_{NMed}	≤25	≥25	=25	≤25	=25
	≤50	≤50	=50	≥50	≥50
	≥25	≤25	=25	=25	≤25

i.e. the situation in which each of the five group members chooses his individual Nash-play, arises in only 2.8% of the social outcomes. Compared to the no info treatment, Nash-play is higher under full information in every period when using peak distributions B, C and D. This outcome is in line with the fact that we observe less truth-telling with full disclosure of the other peaks. Figures 8 and 9 summarize truth-telling, Nash-play and tendency to Nash-play shares of total possible votes over periods and for each peak distribution under no and full information.

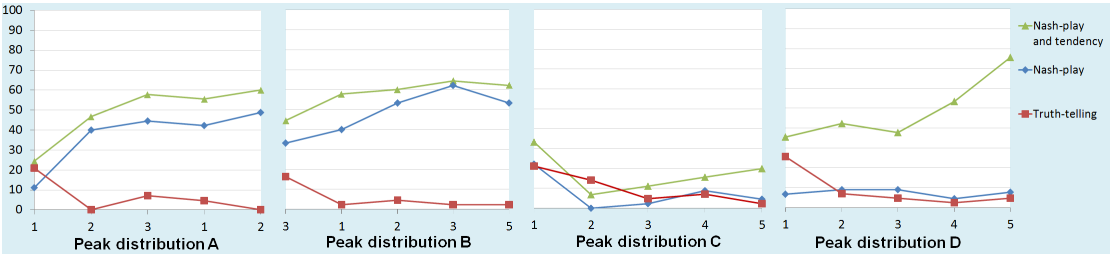


Figure 8: Results of the mean rule under no information (in percentage of total decisions)

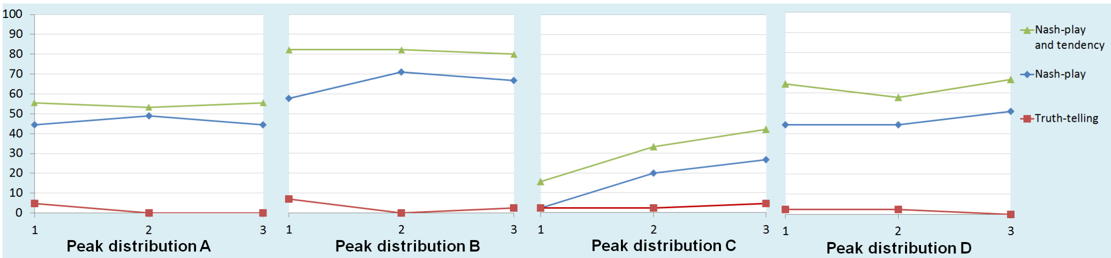


Figure 9: Results of the mean rule under full information (in percentage of total decisions)

A further interesting result can be found with peak distribution C. We created a situation at which all peaks allocate the highest amount on the third project. Only a small proportion of the overall votes are Nash-play (10.8% in comparison to the other three peak distributions with an average of 44.0%) and even under full information, the share is with 16.3% relatively low. Another conspicuousness of peak distribution C is the result that Nash-play decreases by periods under no information. We run a regression of the 'Nash-Vote-Distance' to get a better insight on the deviation from votes to the Nash-play. Table 4 reveals the regression results.

Table 4: Regression results mean: Nash-vote-distance

VARIABLES	(1) NashVoteDist	(2) NashVoteDist
per	-0.854	-0.845
	(1.397)	(1.376)
subj	-0.173	-0.131
	(0.316)	(0.337)
round	-2.032***	-2.048***
	(0.541)	(0.539)
PeakNashDist	0.376***	0.393***
	(0.0524)	(0.0516)
PeakVoteDist	0.485***	0.503***
	(0.0414)	(0.0409)
PPR_Dist	-0.0644***	-0.0664***
	(0.0188)	(0.0186)
edgetruth_d	17.04***	16.65***
	(4.952)	(4.886)
peak_d	34.31***	33.86***
	(4.891)	(4.901)
info_d	2.933	2.944
	(7.651)	(7.613)
male_d		-17.82***
		(3.513)
understandrule_d		-7.529**
		(3.418)
econstudy_d		-5.202*
		(3.114)
Constant	23.78***	44.61***
	(5.876)	(6.492)
Observations	1,440	1,440
R-squared	0.217	0.245

Standard errors in parentheses
*** $p<0.01$, ** $p<0.05$, * $p<0.1$

The coefficients of period ('per') and 'round' have the anticipated negative sign but only the one of the latter variable is statistically significant. This supports the expectation of a learning effect in playing the Nash-strategy over the session. Another indicator for a change in voting behavior is the negative and significant coefficient of the distance between the peak and the result of the previous period ('PPR_Dist'), which reflects the higher gain in utility by Nash-play if the peak is distant from the social outcome of the last round. The positive and significant correlation of the deviation of the vote to the theoretical Nash-play and the peak-Nash-distance ('PeakNashDist') highlights the growing difficulties of finding the corresponding Nash equilibrium the more remote Nash-play is from the peak. The distance between Nash-play and vote increases with a higher peak-vote-distance ('PeakVoteDist'), indicating that manipulation occurs for subjects with a

Nash-play that is more 'difficult' to predict but not towards the Nash equilibrium. We also find a higher deviation from Nash-play if the theoretical Nash-strategy is to vote zero for one project (edge) or to vote for the actual peak (truth) compared to voting zero for two projects, denoted by the positive coefficient of the dummy-variable 'edgetruth_d'. Votes deviate also more from Nash-play if the peak distribution in C, compared to the other peak distributions, see the coefficient of the dummy-variable 'peak_d'. Surprisingly, we are not able to find a significant effect of the degree of information on the distance to Nash-play. Only when including dummy variables with demographic data from the questionnaire, the coefficient of the dummy variable 'info_d' is negative. Nevertheless, we are able to reject the hypothesis that the mean of votes that are Nash-play is similar for rounds without any and under full information. Instead, we find that the mean share of Nash-play is higher under full information compared to no information.

Best Response to the Previous Period Result

When analyzing the voting behavior, we also find some further interesting strategies. Besides truth-telling or Nash-play, subjects might vote for an allocation that is a best response to the social outcome of the previous period result (BRP). Therefore, we first calculate the theoretical best response (tBRP) in period t, where t takes the values 2 to 5 in the no info and 2 and 3 in the full info treatment. Given the votes of the other subjects in the previous period, q_{t-1}^{-i}, under the assumption that the mean of the other subjects remains the same in the current period t, the theoretical best response is to vote for an allocation such that the social outcome is equal to the own peak:

$$p^i = Mean_{t-1}(q) - \frac{1}{5} \cdot q_{t-1}^i + \frac{1}{5} \cdot tBRP_t^i. \tag{6}$$

Solving for the theoretical best response to the previous period result gives the following optimization problem:

$$tBRP_t^i = 5 \cdot (p^i - Mean_{t-1}(q)) + q_{t-1}^i. \tag{7}$$

A nice feature of this computation is the fact that the sum of $tBRP_t^i$ over all j is equal to the total budget $E = 100$. But there might exist allocations that include project-wise votes we prohibited in the experiment, like non-negative votes or ones that exceed the total budget. Hence, by using the tBRP, we calculate the BRP, again, separately for all j projects, but only with feasible allocations $\in \mathcal{B}$ by 'cutting off' the unfeasible ones:

$$BRP_{j,t}^i := \begin{cases} 100, & if\ tBRP_{j,t}^i > 100 \\ 0, & if\ tBRP_{j,t}^i < 0 \\ tBRP_{j,t}^i & else. \end{cases} \tag{8}$$

This calculation excludes prohibited votes, i.e. the allocation for each project is a natural number between zero and 100, like we demanded in the experiment. However, there might be allocations at which the sum of BRP_t^i over all projects exceeds the total budget. Note that by construction of $BRP_{j,t}^i$, it is not possible that the sum of BRP_t^i

over all projects undercuts the total budget. Since the sum of $tBRP_t^i$ over all j is equal to the total budget E, the positive values have to add up to at least E. Thus, in a further step, we create ranges of best responses to the previous period result that reach from a lower (lBRP) to an upper boundary of the BRP (uBRP). Within these ranges, the social outcomes result in equal payoffs due to the hexagon shaped indifference curves. The ranges for each project are calculated by the following equations:

$$lBRP_{j,t}^i = \begin{cases} 100 - BRP_{-j,t}^i, & if\ BRP_{j,t}^i \neq 0 \\ BRP_{j,t}^i & else \end{cases} \tag{9}$$

$$uBRP_{j,t}^i = BRP_{j,t} \tag{10}$$

We find that with 46.9% a relatively high share of votes are indeed a best response to the result of the previous period. Over all peaks and degrees of information, the share of BRP is higher in the last period compared to the first. This result complies with the observation of the increase in Nash-play over time. Once a Nash equilibrium is reached, the BRP in the next period is always equal to the Nash-play, indicating the stability of the Nash equilibrium. When distinguishing between the different info treatments and peak distributions, the results are similar to Nash-play: The share of votes that are a BRP is higher under full information and peak distribution C has a considerably lower share of BRP votes compared to the other distributions. Nevertheless, within peak distribution C the share of votes that are a BRP amount to 27.8% which constitutes a higher share than Nash-play with 10.8%. Therefore, we conclude that even if the theoretical Nash equilibrium is elusive, a considerable share of votes is strategical.

Best Response to a Uniform Distribution of the Other Votes
One might argue that without any information, a reasonable belief on the other subjects' peaks is a uniform distribution on the feasible allocations. This results in an expected mean of the other peaks that allocates a third of the budget on every project. A strategy might now be to play a best response to uniformly distributed votes of the other subjects (BRUD). Table 5 shows all four peaks at which Nash-play and BRUD differ. As the conditions of Nash-play and BRUD are identical for the other peaks, the results are similar. The total share of decision that are a BRUD adds up to 33.5%.[10]

In all cases where BRUD is unequal to Nash-play, five votes are a BRUD (three at peak distribution C and two at D), which represents 8.3% of possible votes. For most peaks Nash-play and BRUD are identical and therefore subjects might play their Nash-strategy automatically because they intend to play a BRUD. Nevertheless, the very low shares of BRUD in cases where it is not identical to Nash-play, show that subjects might indeed realize their Nash-play and not vote according to a BRUD.

3.3.2 The Median Rule

Truth-telling
Compared to the mean rule, the normalized median is more difficult to understand

[10] Due to the restriction on natural numbers of the votes, we considered for peak distribution B all feasible combinations of votes: $q^i = (66, 17, 17)$, $q^i = (67, 16, 17)$ and $q^i = (67, 17, 16)$.

Table 5: Distinctions between Nash-play and BRUD under the mean rule

Peak-distribution	p^i	**Nash-play**	**BRUD**	**#Nash-play**	**#BRUD**
	40	0	$66.\overline{6}$		
B	30	50	$16.\overline{6}$	15	0
	30	50	$16.\overline{6}$		
	15	0	0		
C	30	100	0	6	1
	55	0	100		
	30	100	0		
C	15	0	0	5	1
	55	0	100		
	25	25	0		
D	50	50	100	11	3
	25	25	0		

and therefore the voting behavior is more subtle and not as straightforward to predict. Hence, as stated in Hypothesis H2.1, we anticipate truth-telling since voting for the true preferred allocation is a weakly dominant strategy under the median rule without adaptation. Nevertheless, only 15.8% of all votes are equal to the true peaks.[11] We also find a tendency of less truth-telling with increasing periods over all peak distributions and degrees of information, i.e. the share of truth-telling is lower in period five than in period one. Interestingly, there are rises in truth-telling in the last periods in a lot of peak distributions. This strategy might reveal failure in manipulation and therefore going back to the original voting behavior in the first round. Some participants state in the questionnaire to vote for an extreme allocation in order to irritate the other voters and benefit in the next period. We are able to reject the hypothesis of equal mean shares of truth-telling under full and no information and find a higher share of truth-telling with a higher level of information over all periods and rounds. Since truth-telling is only a weakly dominant strategy in the median rule without adaptation and within-rank deviation of the non-pivotal voters has no effect on the social outcome, the low shares of true votes come not as a surprise.

Best Response to Truth

In a further step, we expand the strategy of truth-telling to a best response to the true peaks of the other participants (BRT). A BRT of an individual, who is pivotal in every project, is to vote for his true preferred allocation. By contrast, the BRT of semi-pivotal or non-pivotal voters is just to stay within their rank, i.e. voting for an equal or higher (lower) value than the pivotal voter if the own true value is higher (lower) in this project. As we assume truth-telling of the other individuals, the BRT might differ from

[11] Again, we only conisder votes where truth-telling is no Nash-play.

the Nash-strategies, where manipulation of the (semi-)pivotal voter might be possible. The detailed BRT of all voters is shown in table 6. An interesting peculiarity is given with respect to peak distributions A and B. The voter that is pivotal in only one project is able to manipulate the social outcome given truth-telling of the other voters. The median of peak distribution A under BRT is $Med(q) = (10, 10, 30)$, resulting in a normalized median of $NMed(q) = (20, 20, 60)$, which is equal to the peak of individual 1. A similar situation holds for peak distribution B, with $Med(q) = (20, 15, 15)$ under BRT, i.e. a normalized median equal to p_1, $NMed(q) = (40, 30, 30)$.

A consideration of the experimental results confirms hypothesis H2.2: a relatively high share of votes (59.7%) are a best response to true revelation of the other peaks. This value fluctuates slightly but remains high over all periods and peak distributions. The shares of BRT are high under both info treatments, we observe 56.0% under no info and 65.9% under full information, and we are able to reject the hypothesis that there is no difference between the treatments. The shares of BRT under full information are significantly higher compared to no information, which stands to reason as the individuals in the experiment can only play a BRT if the entire peak distribution is disclosed.

Nash-play

Table 3 provides the peak distributions we used in the experiment with the corresponding Nash equilibria of the normalized median rule. A comparison to table 6 reveals the differences to best response to truth-telling of the other subjects. While the BRT of semi-pivotal voters is to stay within their rank, Nash-play corresponds to truth-telling. Therefore, the possibility of votes that are a theoretical Nash-play is more limited compared to BRT. Since pivotal voters are able to manipulate the social outcome, Nash-play of the other voters takes manipulation into consideration and adjusts the margins of the ranks, as in peak distributions A and B.

The proportion of votes that are Nash-play amounts to a total of 40.6% and remains constant at a level between 34 and 36% over periods under no information. Under full information, Nash-play decreases unexpectedly from 55.0% in period one to 46.1% in the third period. An analysis of Nash-play under different information treatments leads to a rejection of the hypothesis of equal mean shares in favor of the alternative hypothesis that the mean percentage of Nash-play under full information exceeds the one under no information. We also examine not only Nash-play but Nash equilibrium outcomes and find that 13.5% of all social choices are Nash outcomes. The highest shares of Nash outcomes are achieved with peak distribution C and D, where the social choices are more often equal to the Nash equilibrium outcomes under full information.

When considering only subjects that have a possibility of manipulation under the normalized median rule, i.e. one subject in peak distribution A and B, we find that they never play their corresponding Nash strategy. We conclude that manipulation is not exploited if it is complex but the high values of total Nash-play and BRT show that subjects vote strategically.

Figures 10 and 11 provide an overview of the votes as percentage of the total possible decisions under the median rule that are a best response to truth-telling of the other

Table 6: BRT median

	Individual				
A	**1**	**2**	**3**	**4**	**5**
	20	10	10	70	5
p^i	20	65	8	10	10
	60	25	82	20	85
	≥ 10	≤ 10	≤ 10	≥ 10	≤ 10
BRT^i	≥ 10	≥ 10	≤ 10	≤ 10	≤ 10
	$= 30$	≤ 60	≥ 60	≤ 60	≥ 60
B					
	40	75	70	20	12
p^i	30	10	15	15	78
	30	15	15	65	10
	$= 20$	≥ 40	≥ 40	≤ 40	≤ 40
BRT^i	≥ 15	≤ 15	≤ 15	≤ 15	≥ 15
	≥ 15	≤ 15	≤ 15	≥ 15	≤ 15
C					
	10	20	20	30	15
p^i	20	10	20	15	30
	70	70	60	55	55
	≤ 20	≥ 20	≥ 20	≥ 20	≤ 20
BRT^i	≥ 20	≤ 20	≥ 20	≤ 20	≥ 20
	≥ 60	≥ 60	$= 60$	≤ 60	≤ 60
D					
	10	50	25	15	25
p^i	20	30	50	60	70
	70	20	25	25	5
	≤ 25	≥ 25	≥ 25	≤ 25	≥ 25
BRT^i	≤ 50	≤ 50	$= 50$	≥ 50	≥ 50
	≥ 25	≤ 25	≥ 25	≥ 25	≤ 25

subjects, Nash-play as well as truth-telling per peak distribution and period.

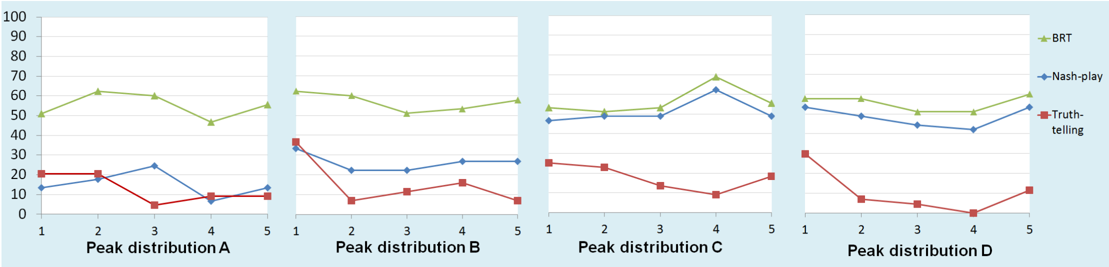


Figure 10: Results of the median rule under no information (in percentage of total decisions)

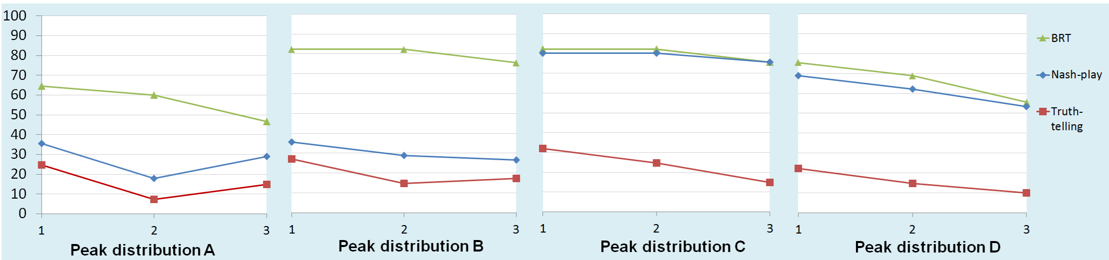


Figure 11: Results of the median rule under full information (in percentage of total decisions)

3.3.3 Mean vs. Median Rule

In order to compare the mean to the median rule, we analyze the influence of the voting rule on the parameters truth-telling, distance between peak and vote as well as Nash-play. We are able to reject the hypothesis of equal mean shares of truth-telling in both voting rules and find a significantly higher share of true votes under the normalized median rule, which confirms hypothesis H3.2. The fact that the median rule is a strategy-proof voting mechanism if we disregard adaptation plays a crucial role in the different voting behaviors. The mean rule is highly manipulable and as mentioned in section 2, in a Nash equilibrium at most one individual votes for a strictly positive amount of every project. By contrast, with the normalized median rule, manipulation is seldom possible and very hard to realize.

Going further into detail, we consider not only truth-telling but also the degree of lying. As stated in hypothesis H3.1, we expect a higher deviation of votes from the true peaks under the mean rule, which is indeed what we find in our analysis. The absolute value of the mean peak-vote-distance under the median rule adds up to 52, as against 32 under the mean rule and the difference is significant.

A comparison between the two voting rules concerning Nash-play is possible, but has to be done cautiously. While the Nash equilibria of the peak distributions we used in the experiment were unique and efficient under the mean rule, there exist several Nash-plays under the median rule that lead to the same social outcome. Moreover, there exist an infinite number of inefficient Nash equilibria, which we excluded in our theoretical analysis. Like stated in section 2, there are ranges of allocations that form a Nash equilibrium and therefore theoretically votes are more often categorized Nash-play under the median rule compared to the mean rule. The experimental results show indeed a higher mean share of Nash-play under the median rule, but since the difference between the rules is not so high, one might argue that the mean rule leads to relatively higher shares. Nevertheless, the mean shares of Nash outcomes are considerably higher under the median rule.

We further analyze the influence of the degree of information on truth-telling and Nash-play under both voting rules. Regarding truth-telling, we are not able to find a significant influence of information on the aggregated results of the mean and the median rule. As stated in the previous sections, the mean share of truth-telling under the median rule is higher with full information, what might come as a surprise. In contrast, under the mean rule, we find a higher mean share of truth-telling when the peaks of the other subjects are unknown.

We are able to reject the hypothesis of equal frequencies of Nash-play under both information treatments. Like already stated before, the mean shares of Nash-play are higher under full information for both voting rules. Comprising results for all voting rules, we also find statistical evidence that full information is accompanied by higher mean shares of Nash-play compared to no information.

4 Conclusion

We developed a theoretical model and determined individual strategies as well as Nash equilibria of the mean and the median rule. The mean rule is highly manipulable and we showed that under single-peaked preferences, at most one individual votes for a strictly positive amount of every public project in a Nash equilibrium. Moreover, we stated that in multi-dimensional allocation problems, inefficient equilibria exist even under the mean rule. Voting behavior under median-based rules are more difficult to predict and manipulation might be possible. In a laboratory experiment, we empirically analyzed the voting behavior under the mean and the median rule on the allocation of three public projects. In particular, we were interested in the occurrence of beneficial manipulation and contrasted both rules.

We observed low shares of truth-telling under the mean rule and a strong tendency of playing the individual Nash equilibrium strategy if it is focal. Nevertheless, group Nash-play rarely emerged. The normalized median rule yielded contrary results. While most subjects played a best response to truth-telling of the other voters, only a small fraction voted truthfully themselves. However, theoretical manipulation was never exploited in the experiment. A comparison of the rule-dependent voting behavior revealed higher shares of truth-telling as well as less absolute deviation from the peak under the median rule. Even though the frequency of Nash-play was higher under the median rule, this result does not provide us with detailed information due to the different Nash-conditions. The findings of the degree of information and its influence on truth-telling are quite surprising. While the shares of truth-telling under the mean rule were higher without any information on the peak distribution, under the median rule, subjects tended to vote truthfully more frequently if information on the detailed peak distribution was provided.

Further research should be done both, on the theoretical model and the experiment. The implementation of manipulation costs as well as further adaptations of the median rule are interesting topics that should be covered.

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