Mobility and Mobility Measures

by

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Abstract

We examine whether mobility indices appropriately represent intergenerational changes in income or status. We suggest three elementary principles for mobility comparisons and show that many commonly-used measures violate one or more them. These principles are used to characterise two classes of indices that have a natural interpretation in terms of distributional analysis.

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1 Introduction

Evidence of individual mobility is often seen as a desirable objective for social and economic policy. It is also used indirectly as part of the discussion of equality of opportunity. Improving data on intra- and inter-generational mobility has greatly advanced the understanding of the strengths and limitations of mobility analysis. However, convincing evidence of mobility requires not only good data but also measurement tools that have appropriate properties. Perhaps surprisingly, several commonly-used techniques and indices do not appear to conform well to simple principles concerning mobility and immobility.

This paper develops some of the ideas in Cowell and Flachaire (2018), to show which types of mobility measures are suitable for the purpose of empirically implementing conventional notions about the meaning of mobility comparisons. We show that there are two broad classes of mobility indices that generally satisfy a minimal set of requirements for mobility comparisons. Each of these classes has a natural interpretation in terms of distributional analysis

The paper is organised as follows. Section 2 sets out some general principles on the meaning of mobility comparisons and examines how well some of the standard tools work in the light of those principles. Section 3 provides a theoretical treatment that embodies the principles of section 2 in a set of axioms and derives a characterisation of two classes of mobility indices from the axioms. Section 4 shows how these broad characterisations can be embodied in two classes of indices that can be easily implemented empirically. Section 5 concludes.

2 Mobility concepts and measures

What do we want a mobility measure to do? Let us discuss this within a very simple context of income change. Assume that there are two periods, labelled 0 and 1, and a given number of individuals with a known status in periods 0 and 1. “Status” could be something very simple, like income, or something derived from the data on the distribution, such as ordinal rank in the distribution. For each person we refer to the pair (status-in-0, status-in-1) as the person’s *history*; if we had intergenerational mobility in mind we might want to refer to the history of a dynasty.

Now let us go through a very short list of principles.

1 More movement, more mobility. There are two principal interpretations of this concept: (1) more movement in one person’s history (or in one dynasty’s history), or (2) more matched movement-in-pairs history. The reason for considering two versions of this principle is that each captures a different concept of mobility. Version (1) allows one to investigate the mobility associated with unbalanced growth: see, for example the discussion in Bourguignon (2011). Version (2) controls for changes in status for a time-1 marginal distribution with given mean; this idea includes the standard interpretations of the concept of “exchange mobility” (Jäntti and Jenkins 2015; Kessler and Greenberg 1981, page 54; McClendon 1977). One or other interpretation of this principle seem to be an almost essential requirement for mobility measurement. The reason is that in each interpretation this principle ensures that a mobility measure has a minimum-mobility property: a situation where there is some movement in status registers higher mobility than a situation where there is a complete absence of movement.

2 Decomposition. Decomposition analysis is routinely applied to other aspects of distributional analysis such as income inequality. Several aspects of decomposability – such as decomposition by population characteristics – seem to be attractive. What has been argued as extremely important is the ability to decompose mobility in terms of upward and downward movements (Bárcena and Cantó 2018).

3 Consistency in comparisons. It is useful for mobility comparisons to have a consistency property. A mobility comparison involves comparing one bivariate distribution of (status-in-0, status-in-1) with another. Suppose one such pair of distributions is clearly "similar" to another – for example where the pair of bivariate distributions in the second case can be found by a simple transformation of the pair of distributions in the first case, perhaps by just rescaling all the status values by a common factor, or by just translating the distributions by increasing/decreasing all the status values by the same given amount.[1] Then the mobility-ranking for the first pair of distirbutions should be the same as for the second pair.

It would be useful to examine whether the tools that are conventionally used to study mobility conform to these three criteria.

2.1 Statistical measures

Many empirical studies use off-the-shelf tools borrowed from statistics and applied econometrics. To investigate these let income be denoted y and assume that status is given by $x = \log(y)$, so that the history of person i, or dynasty i, consists of a log- income pair (x_{0i}, x_{1i}). There are two standard "statistical" meaures that are in wide usage.

2.1.1 The elasticity coefficient

Perhaps the most commonly used measure of mobility is $1 - \hat{\beta}$, where $\hat{\beta}$ is an elasticity coefficient, computed as the OLS estimation of the slope coefficient from a linear regression of log-income in period 1 (x_1) on log-income on period 0 (x_0):

$$x_{1i} = \alpha + \beta x_{0i} + \varepsilon_i. \tag{1}$$

A high value of $1 - \beta$ is usually taken as evidence of significant mobility. However, it is easy to show that a low value does not necessarily imply low mobility. Indeed we can provide many examples for which we have $1 - \hat{\beta} = 0$ but where common sense suggests that there is indeed mobility in log incomes. To see this note that, since

$$\hat{\beta} = \frac{cov(\mathbf{x}_0, \mathbf{x}_1)}{var(\mathbf{x}_0)}, \tag{2}$$

[1] Note that this consistency-in-comparisons property does not imply that the value of a mobility index should be constant under scale or translation changes.

it is true that

$$1-\hat{\beta}=0 \quad \Leftrightarrow \quad cov(\mathbf{x}_0,\mathbf{x}_1)=var(\mathbf{x}_0). \tag{3}$$

It follows that, with equidistant log-incomes $\mathbf{x}_0=(x_{01},x_{01}+k,x_{01}+2k)$ in period 0, and with log-incomes $\mathbf{x}_1=(x_{11},x_{12},x_{11}+2k)$ in period 1, we have $1-\hat{\beta}=0$, $\forall x_{01},x_{11},x_{12}$.[2]

For instance, with log-incomes $\mathbf{x}_0=(1,2,3)$ in period 0, and with log-incomes

$$\mathbf{x}_1 \in \{(2,0,4),(2,1,4),(2,1760,4),(2100,1,2102),(2100,74,2102),\dots\} \tag{4}$$

in period 1, the mobility index based on the elasticity coefficient suggests no mobility: in all cases, we have $1-\hat{\beta}=0$. This implies that the regression coefficient violates the minimal-mobility property discussed under "more movement, more mobility".

2.1.2 The correlation coefficient

As a further problematic example consider another widely used mobility measure, $1-\hat{\rho}$, where $\hat{\rho}$ is the Pearson correlation coefficient. This measure is both scale and translation independent, that is:

$$\text{if} \quad x_1=ax_0+b, \quad \text{then} \quad \hat{\rho}=1 \quad \Leftrightarrow \quad 1-\hat{\rho}=0 \tag{5}$$

So if $x_{1i}=ax_{0i}+b$ across individuals or dynasties we will find that $\rho=1$ so that mobility $(1-\rho)$ is zero. For instance, with log-incomes $\mathbf{x}_0=(1,2,3)$ in period 0 and $\mathbf{x}_1=(0,2,4)$ in period 1, we have $x_1=2x_0-2$ and, thus, $1-\hat{\rho}=0$.

This measure may appear attractive. However, it behaves strangely in several cases. Indeed, we can show that with equidistant log-incomes $\mathbf{x}_0=(x_{01},x_{01}+k,x_{01}+2k)$ in period 0, and with period-1 log-incomes $\mathbf{x}_1=(x_{11},x_{12},x_{11})$, we have $1-\hat{\rho}=1$ and $1-\hat{\beta}=1$, $\forall x_{01},x_{11},x_{12}$.[3]

For instance, with period-0 log-incomes $\mathbf{x}_0=(1,2,3)$ and period-1 log-incomes

$$\mathbf{x}_1 \in \{(3,2,3),(3,0,3),(3,100,3),(1,2,1),(10,1,10),(2,1,2),(2,100,2),\dots\} \tag{6}$$

the standard statistical mobility indices indicate that there is identical mobility: in all cases, we have $1-\hat{\rho}=1$ and $1-\hat{\beta}=1$.

2.1.3 A simple example

Consider the cases depicted in Table 1. It is clear that case 2 exhibits more income movements. Indeed, from period 0 to 1, log-income variations $(\mathbf{x}_1-\mathbf{x})$ are equal to (2,0,0) in case 1 and to (2,-1,2) in case 2. However, with standard mobility measures, $1-\hat{\rho}$ and $1-\hat{\beta}$, we find more mobility in case 1. The measure based on elasticity even suggests no mobility in case 2.

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[2] With equidistant values, mean log-income in period 0 is $\bar{x}_0=x_{01}+k$ and we have $x_{01}-\bar{x}_0=\bar{x}_0-x_{03}=-k$. So $cov(\mathbf{x}_0,\mathbf{x}_1)=var(\mathbf{x}_0)$ is equivalent to $(x_{01}-\bar{x}_0)[(x_{11}-\bar{x}_1)-(x_{13}-\bar{x}_1)]=2(x_{01}-\bar{x}_0)^2$, which can also be written $x_{11}-x_{13}=-2k$.

[3] With equidistant values, mean log-income in period 0 is $\bar{x}_0=x_{01}+k$ and we have $x_{01}-\bar{x}_0=\bar{x}_0-x_{03}$. Thus, $cov(\mathbf{x}_0,\mathbf{x}_1)=k(x_{13}-x_{11})$. When $x_{13}=x_{11}$ we have $cov(\mathbf{x}_0,\mathbf{x}_1)=0$; therefore $\hat{\beta}=\hat{\rho}=0$

	period 0	period 1	mobility	
	$\mathbf{x}_0$	$\mathbf{x}_1$	$1-\hat{\rho}$	$1-\hat{\beta}$
case 1	$(1,2,3)$	$(3,2,3)$	1.0	1.0
case 2	$(1,2,3)$	$(3,1,5)$	0.5	0.0

Table 1: "statistical" mobility measures

2.2 Other mobility indices

Now let us consider other approaches in the literature. In addition to the well-known mobility measures based on the elasticity and correlation coefficients discussed in the introduction, we consider the following mobility measures:

- Fields and Ok (1996) provided a measure of mobility based on income differences:[4]

$$FO_1 = \frac{1}{n}\sum_{i=1} \mid y_{0i} - y_{1i} \mid .$$

- Fields and Ok (1999) provided a measure of mobility based on differences in log incomes:[5]

$$FO_2 = \frac{1}{n}\sum_{i=1} \mid \log y_{1i} - \log y_{0i} \mid .$$

- Shorrocks (1978) provided mobility measures related to inequality:

$$S_I = 1 - \frac{I(y_0+y_1)}{\frac{\mu_{y_0}}{\mu_{y_0+y_1}} I(y_0) + \frac{\mu_{y_1}}{\mu_{y_0+y_1}} I(y_1)},$$

  where $I(.)$ is a predefined inequality measure.

Table 2 presents values of these mobility measures in different situations. We consider a three-person world (A, B, C), with always the same incomes in period 0, $y_0 = (e, e^{1.5}, e^2)$, and several scenarios in period 1, with shifted, rescaled and/or reranked incomes. Elasticity and correlation coefficients are independent of units of measurement of the variables. So mobility indices based on these coefficients respect the scale-independence property. It is clear from Table 2, where scenario 1^a gives a zero value ($1-\beta = 0$), and scenarios 1^c and 1^d provide the same value ($1-\beta = 1.5$). Furthermore, the major drawback provided in the introduction is also clear, since zero mobility is obtained with scenarios 1^f or 1^g. It follows that a low value of these measures cannot be associated to low mobility.

The Fields-Ok mobility measures are not scale-independent, they have values different from zero in scenario 1^a ($FO_1 = 4.863$ and $FO_2 = 0.693$) and they have different values in 1^c and 1^d. In Table 2, we can see that the same value is given to Fields-Ok measures in scenarios 1^f and 1^g, who share the same income values, with the same ranking at the two periods in 1^g and a reranking in 1^f.

[4]No mobility is defined when incomes at both periods are shifted by the same value.

[5]No mobility is defined when incomes int both periods are multiplied by the same value.

	period	period						
	0	1^a	1^b	1^c	1^d	1^e	1^f	1^g
A	$e^{1.0}$	$2\,e^{1.0}$	$e^{1.0}+2$	$e^{1.5}$	$2\,e^{1.5}$	$e^{1.5}+2$	e^1	e^1
B	$e^{1.5}$	$2\,e^{1.5}$	$e^{1.5}+2$	$e^{2.0}$	$2\,e^{2.0}$	$e^{2.0}+2$	e^3	e^2
C	$e^{2.0}$	$2\,e^{2.0}$	$e^{2.0}+2$	$e^{1.0}$	$2\,e^{1.0}$	$e^{1.0}+2$	e^2	e^3
Income-mobility measures								
Elasticity	$1-\beta$	**0**	0.312	*1.500*	*1.500*	1.318	**0**	-1.000
Pearson Corr.	$1-r$	**0**	0.001	*1.500*	*1.500*	1.461	0.500	**0**
Fields-Ok 1	FO_1	4.863	2.000	3.114	6.165	3.781	*5.201*	*5.201*
Fields-Ok 2	FO_2	0.693	0.387	0.667	0.898	0.686	*0.500*	*0.500*
Shorrocks	S_{Theil}	**0**	0.031	0.743	0.679	0.751	0.281	0.069
Shorrocks	S_{Gini}	**0**	**0**	*0.500*	0.459	*0.500*	0.132	**0**

Table 2: Income and rank mobility measures in different scenarios.

The Shorrocks measures are not scale-independent (scenarios 1^c and 1^d provide different values). In addition, they are sensitive to the choice of the inequality index. Indeed, Table 2 gives very different results with the Theil and Gini indices (S_{Theil}, S_{Gini}). At first sight, the Shorrocks index based on the Gini may appear to be an appropriate measure of rank mobility (Aaberge et al. 2002), because it is equal to zero when no individual position shifts takes place (scenarios $1^a, 1^b$ and 1^g). However, it should not be used to measure rank mobility, because two similar reranking scenarios (1^c and 1^d) give different values of the index (0.5 vs 0.459).

3 Mobility measures: theory

In this section we develop the general principles discussed in section 2 to formalise the principles into axioms and then using the axioms to characterise mobility orderings. As in section 2 we deal with the problem of two-period mobility and a fixed population.

3.1 Status, histories, profiles

The fundamental ingredient is the individual's status, which could be income, position in the distribution, or something else. Let u_i denote i's status in period 0 and v_i denote i's status in period 1. Individual *history* is the pair $z_i = (u_i, v_i)$. Individual movements or changes in status are completely characterised by the histories z_i, $i = 1, 2, ..., n$ Call the array of such histories $\mathbf{z} := (z_1, ..., z_n)$ a *movement profile*[6] and denote the set of all possible movement profiles for a population of size n as Z^n.

The principal problem concerns the representation of the changes embedded in a movement profile. Overall mobility for any member of Z^n can be described in terms

[6]note that the profile concept here is somewhat different from that developed in profilesVan Kerm (2009)

of the status changes of each the n persons or dynasties. We need to specify a set of axioms for comparing the elements of Z^n that capture the the principles in section 2

3.2 Mobility ordering: basic structure

In this section and section 3.3 we characterise an ordering that enables us to compare movement profiles. Use $\succeq$ to denote a weak ordering on Z^n; denote by $\succ$ the strict relation associated with $\succeq$ and denote by $\sim$ the equivalence relation associated with $\succeq$. We first consider the interpretation of five axioms that underpin the approach; we then state a basic result that follows from them.

Axiom 1 [Continuity] *$\succeq$ is continuous on Z^n.*

Axiom 2 [Monotonicity] *If $\mathbf{z}, \mathbf{z}' \in Z^n$ differ only in their ith component and $u_i' = u_i$ then, if $v_i > v_i' \geq u_i$, or if $v_i < v_i' \leq u_i$, $\mathbf{z} \succ \mathbf{z}'$*

Axiom 3 [Independence] *Let $\mathbf{z}(\zeta, i)$ denote the profile formed by replacing the ith component of $\mathbf{z}$ by the history $\zeta \in Z$ and let $\hat{Z}_i := \left[u_{(i-1)}, u_{(i+1)}\right] \times \left[v_{(i-1)}, v_{(i+1)}\right]$ where the subscript $(i-1)$ (resp. $(i+1)$)denotes the largest (resp. smallest) component of the vector that is less than (resp. greater than) the ith component. For $\mathbf{z}, \mathbf{z}' \in Z^n$ suppose that $\mathbf{z} \sim \mathbf{z}'$ and $z_i = z_i'$ for some $i \in 2, ..., n-1$: then $\mathbf{z}(\zeta, i) \sim \mathbf{z}'(\zeta, i)$ for all $\zeta \in \hat{Z}_i$.*

Axiom 4 [Local immobility] *Let $\mathbf{z}, \mathbf{z}' \in Z^n$ where for some i, $u_i = v_i$, $v_i' = u_i'$ and, for all $j \neq i$, $u_j' = u_j$, $v_j' = v_j$. Then $\mathbf{z} \sim \mathbf{z}'$.*

.

We can then show:[7]

Theorem 1 *Given Axioms 1 to 4 then $\forall \mathbf{z} \in Z^n$ the mobility ordering $\succeq$ is representable by an increasing monotonic transform of*

$$\sum_{i=1}^{n} \phi_i(z_i), \tag{7}$$

where the ϕ_i are continuous functions $Z \to \mathbb{R}$, defined up to an affine transformation, each of which is increasing (decreasing) in v_i if $v_i > (<)\ u_i$ and that has the property $\phi_i(u, u) = b_i u$, where $b_i \in \mathbb{R}$.

The interpretation of these axioms and the result in Theorem 1 is as follows. First, suppose we know that, with the sole exception of person i, each person's history in profile $\mathbf{z}$ is the same as it is in profile $\mathbf{z}'$. Person i's history can be described as follows: i starts with the same period-0 status in $\mathbf{z}$ and in $\mathbf{z}'$ and then moves up to a higher status in period 1; but i's period-1 status in profile $\mathbf{z}$ is even higher than it is in $\mathbf{z}'$. Then Axiom 2 implies that mobility would be higher in $\mathbf{z}$ than in $\mathbf{z}'$; a corresponding story holds for

[7] The proof is in the Appendix.

downward movement. Second, suppose that the profiles $\mathbf{z}$ and $\mathbf{z}'$ are equivalent in terms of overall mobility and that there is some person i with the same history z_i in both $\mathbf{z}$ and $\mathbf{z}'$. Then consider a change Δz_i in i's history in both $\mathbf{z}$ and $\mathbf{z}'$ that is sufficiently small as to leave unchanged the person's ranking in the initial and final distributions. Axiom 3 requires that such a change leaves the two modified profiles as equivalent in terms of overall mobility. It is this property that allows some form of decomposability of mobility measures. Third, consider a profile $\mathbf{z}$ in which person i is immobile: if i's status by the same amount in both periods then the new profile $\mathbf{z}'$ should exhibit the same mobility as the original $\mathbf{z}$. Theorem 1shows that mobility comparisons should be based on the sum of evaluations of the n individual histories; but these evaluations may differ from person to person and may, accordingly depend on the rank-order of the person's history in the movement profile.[8]

3.3 Mobility ordering: scale

The second part of our characterisation of the mobility ordering involves the comparison of profiles at different levels of status. To do this let $\mathbf{z} \times (\lambda_0, \lambda_1)$ denote the movement profile that is derived from $\mathbf{z}$ if all the 0-components (u_i) are multiplied by λ_0 and all the 1-components (v_i) are multiplied by λ_1. Then we can introduce the following additional axiom:

Axiom 5 [Status scale irrelevance] *For any* $\mathbf{z}, \mathbf{z}' \in Z^n$ *such that* $\mathbf{z} \sim \mathbf{z}'$, $\mathbf{z} \times (\lambda_0, \lambda_1) \sim \mathbf{z}' \times (\lambda_0, \lambda_1)$, *for all* $\lambda_0, \lambda_1 > 0$.

We then have[9]

Theorem 2 *Given Axioms 1 to 5* $\succeq$ *is representable by (7), where* ϕ_i *is given by*

$$\phi_i (u, v) = c_i \left[u^\alpha v^{1-\alpha} - \alpha u - [1 - \alpha] v \right], \tag{8}$$

or by

$$\phi_i (u, v) = a_i [b_i v - u], \tag{9}$$

where $\alpha, a_i b_i, c_i \in \mathbb{R}$.

The interpretation of Axiom 5 is that the ordering of profiles remains unchanged by a scale change to status in either or both periods. As we will see in section 4 it will also allow one to introduce the idea of translation irrelevance, where the ordering of profiles does not change under the across-the-board addition of some constant to status. Theorem 2 shows that the evaluation function ϕ_i in (7) has to take one of the two particularly convenient forms (8) and (9).

[8]This is a slight generalisation of Theorem 1 in Cowell and Flachaire (2018).

[9]Again, the proof is in the Appendix.

3.4 Aggregate mobility index

Theorem 2 means that the mobility ordering $\succeq$ implied by the five axioms in sections 3.2 and 3.3 can be represented by the expression $\sum_{i=1}^{n} \phi_i(u, v)$, with the ϕ_i given by equation (8) or (9). Since $\succeq$ is an ordering it is also representable by some continuous increasing transformation of this expression. We now examine what normalisation is appropriate in order to construct an aggregate inequality index, for each of the two cases in Theorem 2.

3.4.1 Class 1: ϕ_i given by equation (8)

First, let us require that mobility should be blind as to individual identity. If the definition of status incorporates all relevant information about an individual, the labelling $i = 1, ..., n$ is irrelevant and anonymity is an innocuous assumption. It simply means that mobility depends only on individual status histories; switching the personal labels from one history to another within a movement profile has no effect on mobility rankings: if a profile $\mathbf{z}'$ can be obtained as a permutation of the components of another profile $\mathbf{z}$, then they should be treated as equally mobile. If so, then all the c_i should be equal and mobility can be represented as a transform of

$$c \sum_{i=1}^{n} \left[u_i^{\alpha} v_i^{1-\alpha} - \alpha u_i - [1-\alpha] v_i \right]. \tag{10}$$

Now consider the effect of population size. A simple replication of profiles $\mathbf{z}$ does not change the essential facts of mobility. Clearly α cannot depend on the size of the population, but the constant c may depend on n. If any profile is replicated r times and the index remains unchanged under replication we have

$$c(n) \sum_{i=1}^{n} \left[u_i^{\alpha} v_i^{1-\alpha} - \alpha u_i - [1-\alpha] v_i \right] = c(nr)\, r \sum_{i=1}^{n} \left[u_i^{\alpha} v_i^{1-\alpha} - \alpha u_i - [1-\alpha] v_i \right].$$

So, to ensure that the representation of $\succeq$ is in a form that is constant under replication, we need to have c proportional to $1/n$. Choosing for convenience the constant of proportionality as $\frac{1}{\alpha[\alpha-1]}$ we may write the index as some transform of this "basic-form" mobility index:

$$\frac{1}{\alpha[\alpha-1]} \left[\frac{1}{n} \sum_{i=1}^{n} u_i^{\alpha} v_i^{1-\alpha} - \alpha \mu_u - [1-\alpha] \mu_v \right], \tag{11}$$

where

$$\mu_u := \frac{1}{n} \sum_{i=1}^{n} u_i, \tag{12}$$

$$\mu_v := \frac{1}{n} \sum_{i=1}^{n} v_i. \tag{13}$$

Notice that (11) is strictly increasing (decreasing) in u_i if $u_i > v_i$ ($u_i < v_i$) and (11) is strictly decreasing (increasing) in v_i if $u_i < v_i$ ($u_i > v_i$); this behaviour is natural in view of monotonicity (Axiom 2). Furthermore it is clear that the basic form (11) has the property that mobility is zero if $v_i = u_i$ for all i. In general, if the normalised mobility index is to have the appropriate "zero" property, it must take the form

$$\psi\left(\frac{1}{n}\sum_{i=1}^{n} u_i^{\alpha} v_i^{1-\alpha} - \theta\left(\mu_u, \mu_v\right), \mu_u, \mu_v\right) \tag{14}$$

where ψ is monotonic in its first argument and has the property that $\psi\left(0, \mu_u, \mu_v\right) = 0$, and where θ is a function that is homogeneous of degree 1 with the property that $\theta\left(\mu, \mu\right) = \mu$.

3.4.2 Class 2: ϕ_i given by equation (9)

Again consider the issue of anonymity. If the constant b_i is the same for all i, then (9) means that individual mobility for each person i is captured simply by $a_i d_i$, where d_i is the weighted difference in status between periods 0 and 1:

$$d_i := bv_i - u_i. \tag{15}$$

The overall mobility index will preserve anonymity if it is written as $\sum_{i=1}^{n} a_i d_{(i)}$ where $d_{(i)}$ denotes the ith component of the vector $(d_1, ..., d_n)$ when rearranged in ascending order: so, except in the case where there is zero individual mobility, $d_{(1)} < 0$ refers to the greatest downward mobility and $d_{(n)} > 0$ to the greatest upward mobility. The principle of monotonicity is preserved if $a_i < 0$ whenever $d_{(i)} < 0$ and $a_i > 0$ whenever $d_{(i)} > 0$. The independence of population size means that the term a_i should be normalised by $1/n$; so up to a change in scale we have the mobility measure

$$\frac{1}{n}\sum_{i=1}^{n} a_i d_{(i)}. \tag{16}$$

3.5 Mean-Normalisation

The mobility indices derived in section 3.4 are consistent with the version (1) of the "More movement, more mobility" principle discussed in section 2. We now consider a modification that will enable us to handle version (2) of that principle.

This can be done by replacing Axiom 2 with the following Axiom 6:

Axiom 6 [Monotonicity-2] *If $\mathbf{z}, \mathbf{z}' \in Z^n$ differ only in their ith and jth components and $u_i' = u_i$, $u_j' = u_j$, $v_i' - v_i = v_j - v_j'$ then, if $v_i > v_i' \geq u_i$ and if $v_j < v_j' \leq u_j$, $\mathbf{z} \succ \mathbf{z}'$.*

Clearly the type of status variation considered in the statement of Axiom 2 will change the mean of u and/or the mean of v; the type of status variation considered in Axiom 6 will leaves these means unaltered. The modified version of monotonicity in Axiom 6 will again ensure that minimal-mobility property is satisfied. Also Axiom 6 is clearly satisfied by the normalised index.

One consequence of using version (2) of the "More movement, more mobility" principle is that it allows for a further step in normalisation of the mobility indices. It may be appropriate that the mobility index remain unchanged under a scale change $\lambda_0 > 0$ in the 0-distribution and under a scale change $\lambda_1 > 0$ in the 1-distribution. This strengthens the scale-irrelevance property (Axiom 5) that we imposed on mobility orderings to *scale-independence* of the resulting mobility index.

Let us examine this development for each of the two classes aggregate mobility indices derived in section 3.4.

Class 1 mobility indices. Setting $\lambda_0 = 1/\mu_u$ and $\lambda_1 = 1/\mu_v$ it is clear that (14) becomes

$$\psi\left(\frac{1}{n}\sum_{i=1}^{n}\left[\frac{u_i}{\mu_u}\right]^{\alpha}\left[\frac{v_i}{\mu_v}\right]^{1-\alpha} - \theta\left(1,1\right), 1, 1\right) = \overline{\psi}\left(\frac{1}{n}\sum_{i=1}^{n}\left[\left[\frac{u_i}{\mu_u}\right]^{\alpha}\left[\frac{v_i}{\mu_v}\right]^{1-\alpha} - 1\right]\right), \quad (17)$$

where $\overline{\psi}\left(t\right) := \psi\left(t, 1, 1\right)$.

Class 2 mobility indices. Clearly we may obtain a mean-normalised version of (16) by dividing a_i by $1/\mu_u$ and setting $b = \mu_u/\mu_v$ to give

$$\frac{1}{n}\sum_{i=1}^{n} a_i \left[\frac{v_{(i)}}{\mu_v} - \frac{u_{(i)}}{\mu_u}\right]. \quad (18)$$

Specific examples of these mean-normalised indices are given in section 4

4 Discussion and examples

Expressions (8) and (9) characterise the bases for two classes of mobility indices. Here we consider the derivation of practical indices from these two bases.

4.1 Class-1 mobility indices

To ensure that the mobility measure M_α is well-defined and non-negative for all values of α and that, for any profile $\mathbf{z}$, M_α is continuous in α, we adopt the following cardinalisation of (17):

$$M_\alpha := \frac{1}{\alpha\left[\alpha - 1\right] n}\sum_{i=1}^{n}\left[\left[\frac{u_i}{\mu_u}\right]^{\alpha}\left[\frac{v_i}{\mu_v}\right]^{1-\alpha} - 1\right], \ \alpha \in \mathbb{R}, \alpha \neq 0, 1, \quad (19)$$

where we have the following limiting forms for the cases $\alpha = 0$ and $\alpha = 1$, respectively

$$M_0 = -\frac{1}{n}\sum_{i=1}^{n}\frac{v_i}{\mu_v}\log\left(\frac{u_i}{\mu_u}\Big/\frac{v_i}{\mu_v}\right), \quad (20)$$

$$M_1 = \frac{1}{n}\sum_{i=1}^{n}\frac{u_i}{\mu_u}\log\left(\frac{u_i}{\mu_u}\Big/\frac{v_i}{\mu_v}\right). \quad (21)$$

Expressions (19)-(21) satisfy the second version of the movement principle (formalised in the monotonicity-2 Axiom 6) and constitute a *class* of aggregate mobility measures that are independent of population size and independent of the scale of status.

Choice of α.

An individual member of the class is characterised by the choice of α: a high positive α produces an index that is particularly sensitive to downward movements and a negative α yields an index that is sensitive to upward movements.

Furthermore, let us consider a sample where every individual's upward mobility is matched by a symmetric downward mobility of someone else ($\forall i, \exists j$ such that $u_j = v_i, v_j = u_i$). In this particular case of (perfect) symmetry between downward and upward status movements, we have $\mu_u = \mu_v$. Then, it is clear from (19) that a high positive α produces an index that is particularly sensitive to downward movements (where u exceeds v) and a negative α yields an index that is sensitive to upward movements (where v exceeds u).[10]

4.2 Class-2 mobility indices

This class focuses on the the aggregation of weighted status differences d_i defined in (15). Consider their behaviour in respect of the two interpretations of the movement principle (Axioms 2 and 6).

4.2.1 Non-normalised status

Using (16) define i^* as the largest i such that $d_{(i)} < 0$. In order to conform to Axiom 2 a_i must satisfy $a_i < 0$ for $i \leq i^*$ and $a_i \geq 0$ otherwise. Consider the simple specification

$$a_i = \begin{cases} -1 & \text{if } i \leq i^* \\ +1 & \text{if } i > i^* \end{cases} \tag{22}$$

Then (16) becomes

$$\Gamma_0 := \frac{1}{n} \sum_{i=1}^{n} \left| d_{(i)} \right| . \tag{23}$$

Suppose equal weight is placed on period-0 and period 1 status (parameter $b = 1$); then, if status is income (23) becomes the FO_1 index discussed in section 2.2 and if status is log-income (23) becomes the FO_2 index.

However, (23) does not fulfil the second interpretation of the movement principle. Take the case where for persons i and j, the distances are $d_i \geq 0$ and $d_j \leq 0$. Now suppose that a change occurs to these distances such that $\Delta d_i > 0$ and $\Delta d_j = -\Delta d_i < 0$: if a mobility index satisfies Axiom 6 then mobility must increase with this change (an increase in movement). But the index (23) remains unchanged. Clearly this problem could be avoided if (22) were replaced by $\phi\left(i - i^* - \epsilon\right)$[11] where ϕ is an increasing function

[10] With symmetric downward/upward mobility, $M_\alpha = \frac{1}{\alpha[\alpha-1]n} \sum_{i=1}^{n} \left(\frac{v_i}{\mu_v} \left[\frac{u_i}{v_i} \right]^\alpha - 1 \right)$.

[11] Monotonicity requires $a_i < 0$ if $i \leq i^*$. From (25) it is clear that if $i^* = 0$ we have $a_i > 0$ for $i = 1, ..., n$; if $i^* = n$ we have $a_i < 0$ for $i = 1, ..., n$.

with $\phi(0) = 0$ and ϵ is a number between 0 and 1. Setting $\epsilon = 1/2$ and normalising ϕ so that it is independent of population size we have

$$a_i = \phi\left(\frac{i}{n} - p - \frac{1}{2n}\right), \tag{24}$$

where $p := i^*/n$ is the proportion of the population with downward-moving histories. If d_i increases for any $i > i^*$ then measured mobility would increase (in accordance with the first interpretation of the movement principle); if, for any i, j where $i > i^* > j$ the individual distances change such that $\Delta d_i > 0$ and $\Delta d_j = -\Delta d_i < 0$,then again measured mobility would increase (in accordance with the second interpretation of the movement principle).

A particularly interesting special case of (24) is where ϕ is linear so that

$$a_i = \frac{i}{n} - p - \frac{1}{2n}, \tag{25}$$

which again satisfies Axiom 6 (monotonicity) for all mobility profiles. The associated mobility measure is given by

$$\Gamma_1 := \frac{1}{n}\sum_{i=1}^{n}\frac{i}{n}d_{(i)} - \left[p + \frac{1}{2n}\right]\mu_d, \tag{26}$$

where μ_d is the mean of the status differences d_i. Notice that

$$\Gamma_1 = 1/2 G + \mu_d\left[\frac{1}{2} - p\right] \tag{27}$$

where G is the absolute Gini coefficient of the status differences d_i:

$$G := \frac{2}{n}\sum_{i=1}^{n}\frac{i}{n}d_{(i)} - \mu_d\frac{n+1}{n} = \frac{1}{2n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}|d_i - d_j|, \tag{28}$$

and that, for the case of symmetric mobility $\Gamma = 1/2 G$. This has a particularly nice interpretation as half the absolute Gini[12] applied to the status differences d_i.

Other types of mobility index based on (25) are discussed in section 4.4 below.

4.2.2 Mean-normalised status

(23) and (26) can be modified to scale-independent versions by replacing the d_i in (15) with[13]

$$d_i = \frac{v_i}{\mu_v} - \frac{u_i}{\mu_u}.$$

[12] Clearly the absolute Gini satisfies Axiom 2 in the case of symmetric mobility and satisfies Axiom 6 for all mobility profiles.

[13] Note that the mean-normalised version of (26) is not proportional to the conventional Gini evaluated over the weighted status differences.

In the case of class-1 mobility measures (in section 4.1 above), a similar modification immediately gives a class of measures that satisfy the second interpretation of the movement principle (Axiom 6). However, in the case of these class-2 mobility measures, mean-normalisation does not change their behaviour in this respect. To see this take, as before, i and j such that $d_i \geq 0 \geq d_j$ and consider $\Delta v_i > 0$ and $\Delta v_j = -\Delta v_i < 0$; this will ensure that $\Delta d_j = -\Delta d_i < 0$ and it is clear that once again this leads to no change in the mobility measure if the weights a_i are given by (22) and an increase in mobility if the weights a_i are given by (25).

4.3 Translation-independent and "intermediate" indices

We can generate a different class of mobility indices just by replacing the status concept. We do this first for a class of indices that are "intermediate" between scale-independent and translation independent indices, using the terminology of Bossert and Pfingsten 1990, Eichhorn 1988. From these we can get translation independent-indices as a limiting case.

4.3.1 "Intermediate" Class-1 mobility indices

If we replace the u and v by $u+c$ and $v+c$ where c is a non-negative constant then (19) will be replaced by

$$\frac{\theta(c)}{n}\sum_{i=1}^{n}\left[\left[\frac{u_i+c}{\mu_u+c}\right]^{\alpha(c)}\left[\frac{v_i+c}{\mu_v+c}\right]^{1-\alpha(c)}-1\right], \alpha(c)\in\mathbb{R}, \alpha(c)\neq 0,1 \tag{29}$$

where $\gamma\in\mathbb{R}, \beta\in\mathbb{R}_+$, the term $\alpha(c)$ indicates that the sensitivity parameter may depend upon the location parameter c and $\theta(c)$ is a normalisation term given by

$$\theta(c) := \frac{1+c^2}{\alpha(c)^2-\alpha(c)}; \tag{30}$$

$\alpha(c)=0$ and $\alpha(c)=1$ there are obvious special cases of (29) corresponding to (20) and (21). For any given value of c then we have an "intermediate" version of the mobility index.

4.3.2 Translation-independent Class-1 mobility indices

By writing

$$\alpha(c) := \gamma+\beta c \tag{31}$$

and analysing the behaviour as $c\to\infty$ we may say more. Consider the main expression inside the summation in (29); taking logs we may write this as

$$\log\left(\frac{1+\frac{v}{c}}{1+\frac{\mu_v}{c}}\right)+\alpha(c)\left[\log\left(1+\frac{u}{c}\right)+\log\left(1+\frac{\mu_v}{c}\right)-\log\left(1+\frac{v}{c}\right)-\log\left(1+\frac{\mu_u}{c}\right)\right]. \tag{32}$$

Using the standard expansion

$$\log(1+t) = t-\frac{t^2}{2}+\frac{t^3}{3}-... \tag{33}$$

and (31) we find that (32) becomes

$$\log\left(\frac{1+\frac{v}{c}}{1+\frac{\mu_v}{c}}\right)+\left[\beta+\frac{\gamma}{c}\right]\left[u+\mu_v-v-\mu_u-\frac{u^2}{2c}-\frac{\mu_v^2}{2c}+\frac{v^2}{2c}+\frac{\mu_u^2}{2c}...\right]. \tag{34}$$

For finite $\gamma, \beta, u, v, \mu_u, \mu_v$ we find that (34) becomes

$$\beta\left[u-\mu_u-v+\mu_v\right] \tag{35}$$

and

$$\lim_{c\to\infty}\theta\left(c\right)=\lim_{c\to\infty}\frac{1+\frac{1}{c^2}}{\left[\beta+\frac{\gamma}{c}\right]^2-\frac{1}{c}\left[\beta+\frac{\gamma}{c}\right]}=\frac{1}{\beta^2}. \tag{36}$$

From (35) and (36) we can see that in the limit (29) becomes

$$M'_\beta := \frac{1}{n\beta^2}\sum_{i=1}^{n}\left[\mathrm{e}^{\beta[u_i-\mu_u-v_i+\mu_v]}-1\right], \tag{37}$$

for any $\beta \neq 0$. Let $q_i := u_i - \mu_u - v_i + \mu_v$ so that (37) can be written

$$\frac{1}{n\beta^2}\sum_{i=1}^{n}\left[\mathrm{e}^{\beta q_i}-1\right]=\frac{1}{n\beta^2}\sum_{i=1}^{n}\left[1+\beta q_i+\frac{1}{2!}\beta^2q_i^2+\frac{1}{3!}\beta^3q_i^3+\frac{1}{4!}\beta^4q_i^4+...-1\right], \tag{38}$$

using a standard expansion. Noting that $\frac{1}{n}\sum_{i=1}^{n} q_i = 0$, the right-hand side of (38) becomes

$$\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{2!}q_i^2+\frac{1}{3!}\beta q_i^3+\frac{1}{4!}\beta^2q_i^4+...\right]. \tag{39}$$

As $\beta \to 0$ it is clear that (39) tends to $\frac{1}{2n}\sum_{i=1}^{n} q_i^2$. So the limiting form of (37) for $\beta = 0$ is

$$M'_0 := \frac{1}{2}\mathrm{var}\left(v_i-u_i\right). \tag{40}$$

Expressions (37) and (40) give the class of translation-independent mobility measures - where mobility is independent of uniform absolute additions to/subtractions from everyone's income.

4.3.3 Translation-independent Class-2 mobility indices

Since Class-2 mobility indices are based on weighted differences (15) it is easy to see that in the case of non-normalised indices where the parameter $b = 1$ all Class-2 mobility measures are translation-independent: they than take the form $\frac{1}{n}\sum_{i=1}^{n} a_i d_{(i)}$ where $d_i = v_i - u_i$.

However, the mean-normalised versions of the Class-2 mobility indices are not translation-independent, except in the special case where $\mu_u = \mu_v$.

4.4 Decomposability

4.4.1 Class-1 mobility indices

Our axioms induce an additive structure for the mobility index, so that that the Class-1 mobility measures in (19)-(21) are clearly decomposable by arbitrary population subgroups.

Let there be K groups and let the proportion of population falling in group k be p_k, the class of scale-independent mobility measures (19) can be expressed as:

$$M_\alpha = \sum_{k=1}^{K} p_k \left[\frac{\mu_{u,k}}{\mu_u}\right]^\alpha \left[\frac{\mu_{v,k}}{\mu_v}\right]^{1-\alpha} M_{\alpha,k} + \frac{1}{\alpha^2-\alpha}\left(\sum_{k=1}^{K} p_k \left[\frac{\mu_{u,k}}{\mu_u}\right]^\alpha \left[\frac{\mu_{v,k}}{\mu_v}\right]^{1-\alpha} - 1\right) \quad (41)$$

for $\alpha \neq 0, 1$, where $\mu_{u,k}$ ($\mu_{v,k}$) is the mean status in period-0 (period-1) in group k, and μ_u, μ_v are the corresponding population means defined in (12), (13) (so that $\mu_u = K^{-1}\sum_{k=1}^{K} p_k\mu_{u,k}$, $\mu_v = K^{-1}\sum_{k=1}^{K} p_k\mu_{v,k}$). In particular, notice that in the case where $u = x$ and $v = \mu_x$, we obtain the standard formula of decomposability for the class of GE inequality indices (Cowell 2011). We have the following limiting forms for the cases $\alpha = 0$ and $\alpha = 1$, respectively

$$M_0 = \sum_{k=1}^{K} p_k \left[\frac{\mu_{v,k}}{\mu_v}\right] M_{0,k} - \sum_{k=1}^{K} p_k \left[\frac{\mu_{v,k}}{\mu_v}\right] \log\left(\frac{\mu_{u,k}}{\mu_u} \Big/ \frac{\mu_{v,k}}{\mu_v}\right) \quad (42)$$

$$M_1 = \sum_{k=1}^{K} p_k \left[\frac{\mu_{u,k}}{\mu_u}\right] M_{1,k} + \sum_{k=1}^{K} p_k \left[\frac{\mu_{u,k}}{\mu_u}\right] \log\left(\frac{\mu_{u,k}}{\mu_u} \Big/ \frac{\mu_{v,k}}{\mu_v}\right) \quad (43)$$

This means, for example, that we may partition the population unambiguously into an upward status group U (for $u_i \leq v_i$) and a downward status group D (for $u_i > v_i$) and, using an obvious notation, express overall mobility as

$$M_\alpha = w^{\mathsf{U}} M_\alpha^{\mathsf{U}} + w^{\mathsf{D}} M_\alpha^{\mathsf{D}} + M_\alpha^{\mathsf{btw}}, \quad (44)$$

where the weights w^{U}, w^{D} and the between-group mobility component M_α^{btw} are functions of the status-means for each of the two groups and overall; comparing M_α^{U} and M_α^{D} enables one to say precisely where mobility has taken place.

We may use this analysis to extend the discussion of the choice of α in section4.1. Consider the upward status group U (for $u_i \leq v_i$) and the downward status group D (for $u_i > v_i$), as defined in (44). From (19) we have[14]

$$M_\alpha^{\mathsf{U}} = M_{1-\alpha}^{\mathsf{D}} \quad (45)$$

It suggests that mobility measurement of upward movements and of symmetric downward movements would be identical with $\alpha = 0.5$ ($M_{0.5}^{\mathsf{U}} = M_{0.5}^{\mathsf{D}}$). Furthermore, mobility measurement of upward movements with $\alpha = 1$ would be identical to mobility measurement of symmetric downward movements with $\alpha = 0$ ($M_1^{\mathsf{U}} = M_0^{\mathsf{D}}$).

[14] More generally, if we generate a "reverse profile" $\mathbf{z}'(\mathbf{z}) := \{z_i' = (v_i, u_i) \mid z_i = (u_i, v_i), i = 1, ..., n\}$ by reversing each person's history – swapping the us and vs in (19) – we have $M_\alpha(\mathbf{z}'(\mathbf{z})) = M_{1-\alpha}(\mathbf{z})$.

In the mobility index M_α, the weights given to upward mobility and to downward mobility can be studied through its decomposability property. With symmetric upward/downward status movements, from (41) and (44), we can see that[15]

1. for $\alpha = 0.5$, we have $w^{\mathsf{U}} = w^{\mathsf{D}}$,
2. for $\alpha < 0.5$, we have $w^{\mathsf{U}} > w^{\mathsf{D}}$,
3. for $\alpha > 0.5$, we have $w^{\mathsf{U}} < w^{\mathsf{D}}$.

In other words, $\alpha = 0.5$ puts the same weight on both upward and downward mobility components in (41), while $\alpha < 0.5$ ($\alpha > 0.5$) puts more weights on upward (downward) mobility component. The sensitivity parameter α enables us to capture *directional sensitivity* in the mobility context:[16] high positive values result in a mobility index that is more sensitive to downward movements from period 0 to period 1; negative α is more sensitive to upward movements. Picking a value for this parameter is a normative choice.

4.4.2 Class-2 mobility indices

As with inequality indices such as the Gini coefficient, exact decomposition of by population subgroups is not usually possible. But for Upward/Downward decompositions of the non-normalised mobility indices in section 4.2 we have easily interpretable results. Clearly (23) can be rewritten as

$$\begin{aligned}\Gamma_0 &= p\frac{1}{np}\sum_{i=1}^{np}[-]d_{(i)} + [1-p]\,\frac{1}{n-np}\sum_{i=1}^{n-np}[+]d_{(i+np)} \\ &= -pd^{\mathsf{D}} + [1-p]\,d^{\mathsf{U}},\end{aligned}$$

where d^{D}, d^{U} are the average weighted distance of downward and upward moves, respectively. Now consider the general class of indices that come from using the weights (24) in (15):

$$\Gamma = \frac{1}{n}\sum_{i=1}^{n}\phi\left(\frac{i}{n} - p - \frac{1}{2n}\right)d_{(i)} \tag{46}$$

If the function ϕ has the property that $\phi(\lambda\theta) = \psi(\lambda)\,\phi(\theta)$ for all real numbers λ, θ and some function ψ then (46) can be written as[17]

$$\Gamma = p\psi(p)\,\Gamma^{\mathsf{D}} + [1-p]\,\psi(1-p)\,\Gamma^{\mathsf{U}} \tag{47}$$

[15] From (41) and (44), we have $w^{\mathsf{U}} = p_1(\mu_{u,1}/\mu_u)^\alpha(\mu_{v,1}/\mu_v)^{1-\alpha}$ and $w^{\mathsf{D}} = p_2(\mu_{u,2}/\mu_u)^\alpha(\mu_{v,2}/\mu_v)^{1-\alpha}$. With symmetric downward/upward mobility, we also have $p_1 = p_2$, $\mu_{u,1} = \mu_{v,2} < \mu_{v,1} = \mu_{u,2}$ and $\mu_u = \mu_v$. Then, $w^{\mathsf{U}}/w^{\mathsf{D}} = (\mu_{u,2}/\mu_{v,2})^{1-2\alpha}$, which is greater (less) than one if $1-2\alpha > (<)0$.

[16] See also : Bhattacharya and Mazumder (2011), Corak et al. (2014), Demuynck and Van de gaer (2010) and Schluter and Van de gaer (2011).

[17] To see this note that $\Gamma = p\psi(p)\frac{1}{np}\sum_{i=1}^{np}\phi\left(\frac{1}{p}\left[\frac{i}{n} - p - \frac{1}{2n}\right]\right)d_{(i)} + [1-p]\,\psi(1-p)\frac{1}{n-np}\sum_{i=np+1}^{n}\phi\left(\frac{1}{1-p}\left[\frac{i}{n} - p - \frac{1}{2n}\right]\right)d_{(i)}$, $\Gamma^{\mathsf{D}} = \frac{1}{np}\sum_{i=1}^{np}\phi\left(\frac{1}{p}\left[\frac{i}{n} - p - \frac{1}{2n}\right]\right)d_{(i)}$ and $\Gamma^{\mathsf{U}} = \frac{1}{n-np}\sum_{i=np+1}^{n}\phi\left(\frac{1}{1-p}\left[\frac{i}{n} - p - \frac{1}{2n}\right]\right)d_{(i)}$.

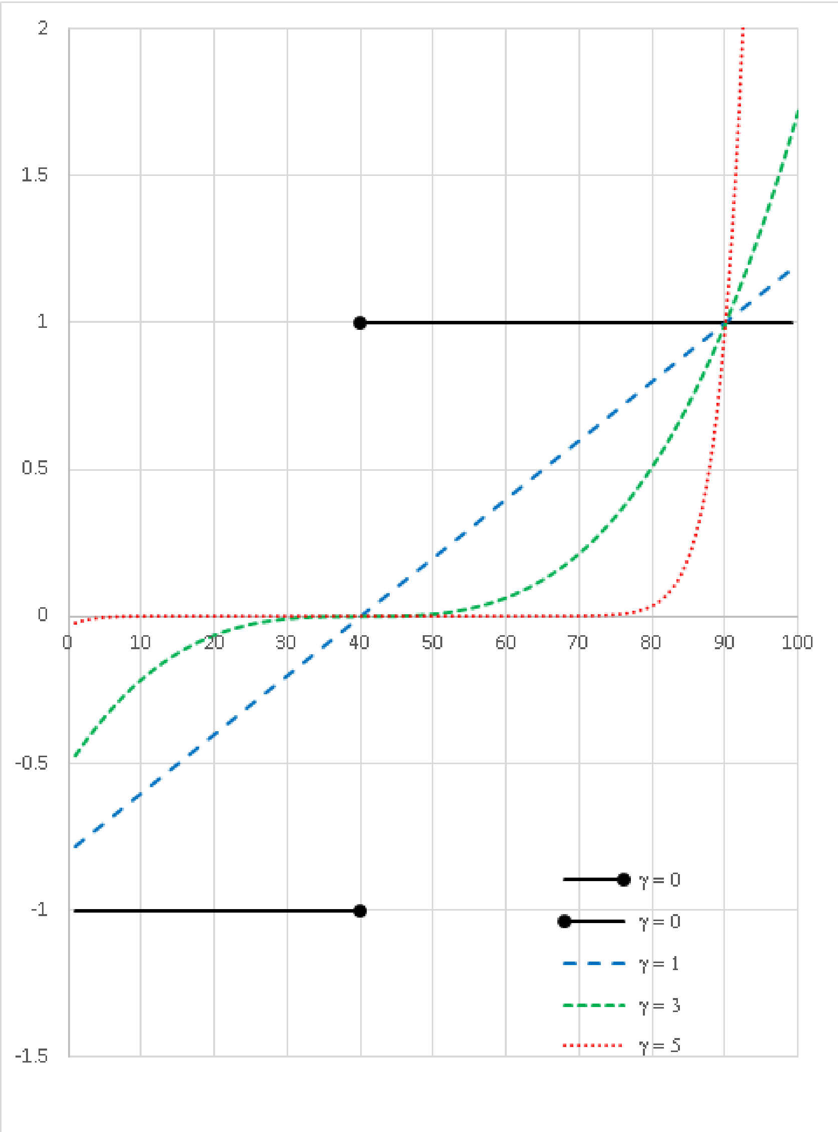

Figure 1: The weights a_i for different values of γ

where Γ^{D} is mobility measured over the set of downward movers

$$\Gamma^{\mathsf{D}} = \frac{1}{np}\sum_{i=1}^{np}\phi\left(\frac{i}{np} - 1 - \frac{1}{2np}\right) d_{(i)}$$

and Γ^{U} is mobility computed over the rest of the population:

$$\Gamma^{\mathsf{U}} = \frac{1}{n-np}\sum_{i=1}^{n-np}\phi\left(\frac{i}{n-np} - \frac{1}{2\left[n-np\right]}\right) d_{(i+np)}.$$

The property (47) requires that ϕ take the form $\phi\left(\theta\right) = A\theta^{\gamma}$ where A and γ are constants. This means that the mobility measure becomes

$$\Gamma_{\gamma} = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{i}{n} - p - \frac{1}{2n}\right]^{\gamma} d_{(i)}, \tag{48}$$

where γ is any odd number greater than or equal to 1, which gives the decomposition formula

$$\Gamma_{\gamma} = p^{\gamma+1}\Gamma_{\gamma}^{\mathsf{D}} + \left[1-p\right]^{\gamma+1}\Gamma_{\gamma}^{\mathsf{U}}. \tag{49}$$

Figure 1 shows the system of weights that emerge in a population of 100 where 40 persons experience downward movement. It is clear that the higher value of γ puts more weight toward the extremes of the distribution of distances d.

5 Conclusion

Mobility measurement deserves careful consideration in the way that the measurement of social welfare, inequality or poverty deserves careful consideration. This consideration should involve principles, formal reasoning and empirical applicability. However, the bulk of empirical studies of mobility analysis apply ready-made techniques that, in this application, are seriously flawed. The flaws matter because, in certain circumstances, the ready-made techniques give exactly the wrong guidance on basic questions such as "does scenario A exhibit more movement of individuals than scenario B?"

Our approach is to show that a consistent theory of mobility measurement can be founded on three basic principles of mobility comparisons. We do this using the methodology of Cowell and Flachaire (2017, 2018) to provide a natural interpretation of these principles in terms of formal axioms. These axioms are used in theorems that characterise two classes of mobility measures that can be easily implemented in terms of income (wealth) mobility or rank mobility.

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Appendix: Proofs

Proof. [Theorem 1].In both the case where Z is a connected subset of $\mathbb{R}\times\mathbb{R}$ and the case where Z is $\mathbb{Q}_+\times\mathbb{Q}_+$ Theorem 5.3 of Fishburn (1970) can be invoked to show that axioms 1 to 3 imply that $\succeq$ can be represented as

$$\sum_{i=1}^{n}\phi_i(z_i),\forall\mathbf{z}\in Z^n, \tag{50}$$

where ϕ_i is continuous, defined up to an affine transformation and, by Axiom 2 is increasing in v_i if $v_i>u_i$ and *vice versa*. Using Axiom 4 in (50) we have

$$\phi_i(u_i,u_i)=\phi_i(u_i+\delta,u_i+\delta), \tag{51}$$

where $\delta:=u_i'-u_i$. Equation (51) implies that ϕ_i must take the form $\phi_i(u,u)=a_i+b_iu$. Since ϕ_i is defined up to an affine transformation we may choose $a_i=0$ and so we have

$$\phi_i(u,u)=b_iu. \tag{52}$$

■

Proof. [Theorem 2]. The proof proceeds by considering two cases of (λ_0,λ_1).

Case 1: $\lambda_0=\lambda_1=\lambda>0$.

Theorem **1** implies that if $\mathbf{z}\sim\mathbf{z}'$ then

$$\sum_{i=1}^{n}\phi_i(z_i)=\sum_{i=1}^{n}\phi_i(z_i'). \tag{53}$$

Axiom 5 further implies that

$$\sum_{i=1}^{n}\phi_i(\lambda z_i)=\sum_{i=1}^{n}\phi_i(\lambda z_i').$$

These two equations imply that the function (7) is homothetic so that we may write

$$\sum_{i=1}^{n}\phi_i(\lambda z_i)=\theta\left(\lambda,\sum_{i=1}^{n}\phi_i(z_i)\right), \tag{54}$$

$$\sum_{i=1}^{n}\psi_i(\lambda v_i)=\theta\left(\lambda,\sum_{i=1}^{n}\psi_i(v_i)\right), \tag{55}$$

where $\theta:\mathbb{R}\to\mathbb{R}$ is increasing in its second argument. Consider the case where, for arbitrary distinct values j and k, we have $v_i=u_i=0$ for all $i\neq j,k$. This implies that $\phi_i(u_i,v_i)=0$ for all $i\neq j,k$ and so, for given values of v_j,v_k,λ, (54) can be written as the functional equation:

$$f_j(u_j)+f_k(u_k)=h(g_j(u_j)+g_k(u_k)), \tag{56}$$

where $f_i(u) := \phi_i(\lambda u, \lambda v_i)$, $g_i(u) := \phi_i(u, v_i)$, $i = j, k$ and $h(x) := \theta(\lambda, x)$. Alternatively, for given values of u_j, u_k, λ, (54) can be written as the functional equation

$$f_j(v_j) + f_k(v_k) = h(g_j(v_j) + g_k(v_k)), \tag{57}$$

with $f_i(v) := \phi_i(\lambda u_i, \lambda v)$, $g_i(v) := \phi_i(u_i, v)$, $i = j, k$ and $h(x) := \theta(\lambda, x)$. Take first the functional equation (56): it has the solution

$$\begin{aligned} f_i(u) &= a_0 g_i(u) + a_i,\ i = j, k, \\ h(x) &= a_0 x + a_j + a_k, \end{aligned}$$

where a_0, a_j, a_k, are constants that may depend on λ, v_j, v_k (Polyanin and Zaitsev 2004, Supplement S.5.5). Therefore:

$$\phi_j(\lambda u_j, \lambda v_j) = a_0(\lambda, v_j, v_k)\phi_j(u_j, v_j) + a_j(\lambda, v_j, v_k) \tag{58}$$

$$\phi_k(\lambda u_k, \lambda v_k) = a_0(\lambda, v_j, v_k)\phi_k(u_k, v_k) + a_k(\lambda, v_j, v_k). \tag{59}$$

Since j and k are arbitrary, we could repeat the analysis for arbitrary distinct values j and ℓ and $v_i = u_i = 0$ for all $i \neq j, \ell$, where $\ell \neq k$; then we would have

$$\phi_j(\lambda u_j, \lambda v_j) = a'_0(\lambda, v_j, v_k)\phi_j(u_j, v_j) + a'_j(\lambda, v_j, v_\ell) \tag{60}$$

$$\phi_k(\lambda u_\ell, \lambda v_\ell) = a'_0(\lambda, v_j, v_k)\phi_\ell(u_\ell, v_\ell) + a'_\ell(\lambda, v_j, v_\ell). \tag{61}$$

where a'_0, a'_j, a_ℓ, are constants that may depend on λ, v_j, v_ℓ. The right-hand sides of (58) and (60) are equal and so a_j must be independent of v_j and a_0 must be independent of v_j, v_k. Therefore, because j and k are arbitrary we have

$$\phi_i(\lambda u_i, \lambda v_i) = a_0(\lambda)\phi_i(u_i, v_i) + a_i(\lambda, v_i),\ i = 1, ..., n. \tag{62}$$

In the case where $v_i = u_i$, (52) and (62) yield

$$b_i \lambda v_i = a_0(\lambda) b_i v_i + a_i(\lambda, v_i)$$

so that

$$a_i(\lambda, v_i) = [\lambda - a_0(\lambda)] b_i v_i$$

and (62) can be rewritten

$$\phi'_i(\lambda u_i, \lambda v_i) - b_i v_i = a_0(\lambda)\phi'_i(u_i, v_i),\ i = 1, ..., n. \tag{63}$$

where $\phi'_i(u_i, v_i) := \phi_i(\lambda u_i, \lambda v_i) - b_i v_i$. From Aczél and Dhombres (1989), page 346 there must exist $\beta \in \mathbb{R}$ and a function $h : \mathbb{R}_+ \to \mathbb{R}$ such that $\phi'_i(u_i, v_i) = u_i^\beta h_i(v_i/u_i)$, so that

$$\phi_i(u_i, v_i) = u_i^\beta h_i\left(\frac{v_i}{u_i}\right) + b_i u_i. \tag{64}$$

From (52) we see that (64) implies $h_i(1) = 0$. Now return to the alternative functional equation (57): following the same argument this must have a solution of the form

$$\phi_i(u_i, v_i) = u_i^{\beta'} h'_i\left(\frac{v_i}{u_i}\right) + b_i v_i \tag{65}$$

Case 2: $\lambda_0 = 1, \ \lambda_1 = \lambda \neq 1.$

Again if $\mathbf{z} \sim \mathbf{z}'$ then (53) holds. Now Axiom 5 implies

$$\sum_{i=1}^{n} \phi_i (u_i, \lambda v_i) = \sum_{i=1}^{n} \phi_i (u_i, \lambda v_i') . \qquad (66)$$

Equations (53) and (66) imply that the function (7) is homothetic in v so that we may write

$$\sum_{i=1}^{n} \psi_i (\lambda v_i) = \theta \left(\lambda, \sum_{i=1}^{n} \psi_i (v_i) \right) , \qquad (67)$$

where $\psi_i (v) := \phi_i (u_i, v)$ and $\theta : \mathbb{R} \to \mathbb{R}$ is increasing in its second argument. By the same argument as Case 1:

$$\psi_i (\lambda v_i) \ = \ a_0 (\lambda) \psi_i (v_i) + a_i (\lambda) , \ i = 1, ..., n. \qquad (68)$$

Putting $v_i = 0$ in (68) implies $a_i (\lambda) = \psi_i (0) [1 - a_0 (\lambda)]$ and (68) becomes

$$\psi_i' (\lambda v) \ = \ a_0 (\lambda) \psi_i' (v) , \text{ where} \qquad (69)$$

$$\psi_i' (v) \ := \ \psi_i (v) - \psi_i (0) . \qquad (70)$$

Equation (69) can be expressed as $f (x + y) = g(y) + f(x)$ where$f (\centerdot) := \log (\psi_i' (\centerdot))$, $g (\centerdot) := \log (a_0 (\centerdot))$, $x = \log v$, $y = \log \lambda$. This Pexider equation has the solution $f (x)$=$bx + c$, $g (y) = \alpha y$

$$\log \psi_i' (v) = a + b \log v, \ \log (a_0 (\lambda)) = b (\log \lambda)$$

where the constant a may depend on i and u_i. This implies

$$\phi_i (u_i, v_i) = A_i (u_i) v_i^b + \phi_i (u_i, 0) . \qquad (71)$$

where $A_i (u_i) = \exp (a)$. Putting $v_i = u_i$ in (64) and (71) we find

$$A_i (u_i) u_i^b + \phi_i (u_i, 0) = b_i u_i.$$

since the RHS is linear in u_i we must have $A_i (u_i)$ proportional to u_i^{1-b}. Therefore

$$\phi_i (u_i, v_i) = c_i v_i^b u_i^{1-b} + \phi_i (u_i, 0) \qquad (72)$$

Now combine the results from the two cases. Since (64), (65) and (72) are true for arbitrary u_i, v_i this implies that

$$\phi_i (u_i, v_i) = c_i u_i^\alpha v_i^{1-\alpha} + c_i' u_i + c''{}_i v_i \qquad (73)$$

where $\alpha := 1 - b$.

From (73) we distinguish two cases

1. $c_i \neq 0$: to ensure that Axiom 2 is satisfied the change in $\phi_i (u_i, v_i)$ must be zero when $v_i = u_i$: this requires $c_i' = -\alpha c_i$ and $c_i'' = - [1 - \alpha] c_i$. This implies (8).

2. $c_i = 0$: to ensure that Axiom 2 is satisfied c_i' and c_i'' must be of opposite sign. This implies (9).

■