An model of social welfare improving transfers

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Introduction

The main challenges of any model of social welfare assessment:

- To define the 'determinants' of welfare (*normative views*),
- To define a 'modification' of the individual outcomes which improves social welfare (*normative views*),
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x socially better than x'

$\Longleftrightarrow$

x obtained from x' by means of social welfare improving transfers

Introduction (2)

The HLP theorem (Hardy et al., 1952), applied to equal mean income distributions, establishes the equivalence between:

(a) x is obtained from x' by a sequence of PD transfers,

(b) $\mathbb{E}[u(x)] \geq \mathbb{E}[u(x')]$, for all u concave,

(c) The Lorenz curve of x lies nowhere below that of x'.

It has been popularized in the *theory of decision under risk* by Rothschild and Stiglitz (1970), and in *inequality measurement* by Kolm (1969) and Atkinson (1970).

This result: a continuous / cardinal / one-dimensional variable. A large literature on *extensions in different frameworks*.

Introduction (3)

The framework:

Social welfare can be *multidimensional*, with *cardinal* or *ordinal* measurable dimensions.

We do not consider ‘specific’ sets of social welfare improving transfers.

We define a general set of transfers on the basis of the *minimal properties* it could satisfy.

Approach compatible with ‘almost all’ the usual transfers considered in the literature: increments, Pigou-Dalton, . . .

Our contribution:

We obtain an *abstract generalization* of the well-known Hardy et al. (1952)’s theorem (a part of).

It can be used to ‘close the loop’ of incomplete HLP-type results . . .

It opens perspectives for ‘new’ definitions of social welfare improvement.

Introduction (4)

The paper builds on two (related) **literature**:

- Theory of majorization (see Marshall et al., 2011),
- Theory of decision under risk (Muller and Scarsini, 2012; Muller, 2013).

Marshall et al. (1967) introduce the notion of *quasi-ordering induced by a convex cone* (unidimensional real-valued variable).

Marshall (1991) investigates, in this context, the equivalence between statements of type (a) and (b) of the HLP theorem.

Muller and Scarsini (2012): extension to multivariate probability distributions with real-valued dimensions $+$ sequence of *inframodular mass transfers* defined as a quasi-ordering induced by a convex cone.

Muller (2013): generalization to any transfers having the same structure.

This paper: a *full discretization* of the last result.

Overview

1. Outcomes, distributions and transfers
2. Social welfare functions
3. Main result
4. Applications

Outcomes, distributions and transfers

1.1. Outcomes and distributions

Set of outcomes: a partially ordered, finite and fixed set $\mathcal{S} \subset \mathbb{Z}_+^d$.

→ *d dimensions* to assess personal's welfare (health, education, ...)

→ each dimension is *finite*, *ordered* and *discrete*

A distribution: a list $n = (n_s)_{s \in \mathcal{S}}$, where $n_s \in \mathbb{Z}_+$ indicates the number of individuals having (d-dimensional) outcome $s \in \mathcal{S}$.

Set of distributions: $\mathcal{N} = \{n \in \mathbb{Z}_+^{|\mathcal{S}|} \mid \sum_{s \in \mathcal{S}} n_s = N\}$, with N *fixed*.

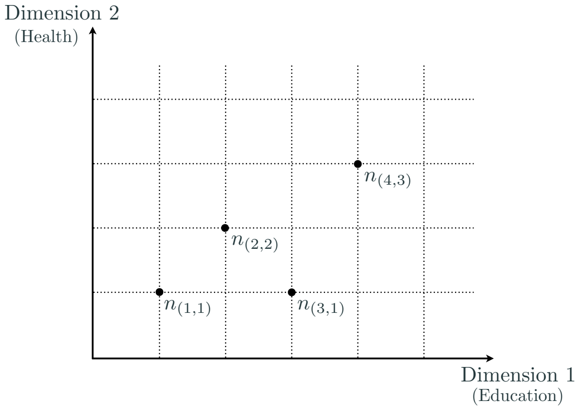
This **discrete framework** is relevant for:

→ *Cardinal* dimensions (income, in euro cents)

→ *Ordinal* dimensions (defined on an ordered categorical scale)

1.1. Outcomes and distributions (2)

Example in *two dimensions*: an outcome is a dot $s = (i, j) \in \mathcal{S} \subset \mathbb{Z}_+^2$, and a distribution is simply a list of $n_{(i,j)}$ for each dot $(i, j) \in \mathcal{S}$.



1.2. Social welfare improving transfers

A **transfer** is here a modification of the distribution which improves the social welfare (according the social planner's views).

Definition 1 (Set of transfers)

The set $\mathcal{T}$ of transfers is the set of all $m \in \mathbb{Z}^{|\mathcal{S}|}$ which satisfies the following two properties:

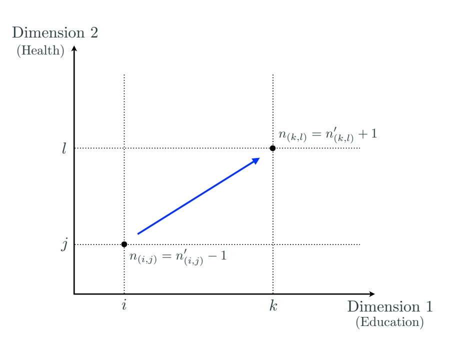
(i) There exist $n, n' \in \mathcal{N}$ such that m can be written as $m = n - n'$,
(a transfer can be written as the difference between two distributions)

(ii) $m \in \mathcal{T}$ and $(-m) \in \mathcal{T}$ imply $m = 0$,
(if it is welfare improving, the reverse transfer is welfare reducing)

Most of the usual transfers considered in the literature are particular cases of $\mathcal{T}$: increments, Pigou-Dalton income transfers, ...

1.2. Social welfare improving transfers (2)

(bidimensional) example: n is obtained from n' by an *increment*, if the distributions are equal everywhere, except in two outcomes $(i,j),(k,l) \in \mathbb{Z}_+^2$ such that $(i,j) < (k,l)$ and:



The *set of increments* $\mathcal{T}_I$ is the set all $m \in \mathbb{Z}^{|\mathcal{S}|}$ such that, either $m = 0$, or there exist two outcomes $(i,j),(k,l) \in \mathbb{Z}_+^2$ with $(i,j) < (k,l)$ such that $m_s = 0$ for all $s \neq (i,j),(k,l)$, $m_{(i,j)} = -1$, and $m_{(k,l)} = 1$.

1.2. Social welfare improving transfers (3)

A **sequence of transfers** is simply a succession of elementary transfers.

Definition 2 (Sequence of transfers)

For all $m, m' \in \mathbb{Z}^{|\mathcal{S}|}$, we write $m \succeq_{\mathcal{T}} m'$ if and only if m can be obtained from m' by means of a linear combination of transfers in $\mathcal{T}$, with positive integer coefficients.

Mathematical interlude:

$m \succeq_{\mathcal{T}} m' \Leftrightarrow (m - m') \in \mathcal{D}(\mathcal{T}) = \{\sum_{t=1}^{|\mathcal{T}|} \lambda_t m_{.t} \mid \lambda_t \in \mathbb{Z}_+ , m_{.t} \in \mathcal{T}\}$.

$\mathcal{D}(\mathcal{T})$ is a *discrete cone*: $m_{.1}, m_{.2} \in \mathcal{D}(\mathcal{T}) \Rightarrow (\lambda_1 m_{.1} + \lambda_2 m_{.2}) \in \mathcal{D}(\mathcal{T})$ for all $\lambda_1, \lambda_2 \in \mathbb{Z}_+$.

$\succeq_{\mathcal{T}}$ is a *quasi-ordering, induced by the discrete cone* generated by $\mathcal{T}$.

Social welfare functions

2.1. SWF consistent with $\succeq_{\mathcal{T}}$

We assume that the social planner's preferences can be represented by a social welfare function $W : \mathbb{Z}^{|\mathcal{S}|} \to \mathbb{R}$.

Definition 3 (Consistency with $\succeq_{\mathcal{T}}$)

We say that W is consistent with $\succeq_{\mathcal{T}}$ if and only if, for all $m, m' \in \mathbb{Z}^{|\mathcal{S}|}$, $m \succeq_{\mathcal{T}} m' \implies W(m) \geq W(m')$. The set of all functions $W \in \mathcal{W}$ consistent with $\succeq_{\mathcal{T}}$ is written $\mathcal{W}_{\mathcal{T}}$.

$\mathcal{W}_{\mathcal{T}}$ encompasses all the social planners sharing the welfare views of transfers in $\mathcal{T}$.

$\mathcal{W}_{\mathcal{T}}$ is a *convex cone*: $W_1, W_2 \in \mathcal{W}_{\mathcal{T}} \Rightarrow (\lambda_1 W_2 + \lambda_2 W_2) \in \mathcal{W}_{\mathcal{T}}$, for all $\lambda_1, \lambda_2 \in \mathbb{R}_+$.

2.2. Utilitarianism

Let $u = (u_s)_{s \in \mathcal{S}} \in \mathbb{R}^{|\mathcal{S}|}$ be a list, assigning utility u_s to outcome $s \in \mathcal{S}$.

$\mathcal{U}$ is the set of all u, defined up to an increasing affine transformation.

A **utilitarian social welfare function** $W_u : \mathbb{Z}^{|\mathcal{S}|} \to \mathbb{R}$ with $u \in \mathcal{U}$, is defined, for all $m \in \mathbb{Z}^{|\mathcal{S}|}$, by:

$$W_u(m) = \sum_{s \in \mathcal{S}} m_s u_s \,.$$

We focus on the utilitarian class for two reasons:

→ this approach is *broadly used* in the literature,

→ our main result identifies it as a *core set of preferences* within the largest class of social welfare functions.

We let $\mathcal{U}_{\mathcal{T}} = \left\{ u \in \mathcal{U} \;\middle|\; \sum_{s \in \mathcal{S}} m_s u_s \geq 0 \,,\; \forall m \in \mathcal{T} \right\}$.

The following result establishes that a utilitarian social welfare function is consistent with $\succeq_{\mathcal{T}}$ if and only if $u \in \mathcal{U}_{\mathcal{T}}$.

Theorem 1

The following two statements are equivalent:

(a) $\forall m, m' \in \mathbb{Z}^{|\mathcal{S}|} : \quad m \succeq_{\mathcal{T}} m' \implies W_u(m) \geq W_u(m')$,

(b) $u \in \mathcal{U}_{\mathcal{T}}$.

The set $\mathcal{U}_{\mathcal{T}} \subset \mathcal{U}$ is a *convex cone*.

Main result

Theorem 2

Let $\mathcal{T}$ be a minimal set of transfers. For all $m, m' \in \mathbb{Z}^{|\mathcal{S}|}$ such that $\sum_{s\in\mathcal{S}} m_s = \sum_{s\in\mathcal{S}} m'_s < \infty$, the following statements are equivalent:

(a) $m \succeq_{\mathcal{T}} m'$,

(b) $W(m) \geq W(m'), \forall\, W \in \mathcal{W}_{\mathcal{T}}$,

(c) $W_u(m) \geq W_u(m'), \forall\, u \in \mathcal{U}_{\mathcal{T}}$.

Important remarks:

→ Statement (a) is a quasi-ordering induced by a *discrete cone*, whereas (b) and (c) are quasi-orderings induced by a *convex cone*.

→ An unanimous ranking within the utilitarian class, for all $u \in \mathcal{U}_{\mathcal{T}}$ is necessary, *but also sufficient* to ensure the unanimity among the larger class $\mathcal{W}_{\mathcal{T}}$.

3.1. Equivalence theorem (2)

To sum up: If $\succeq_{\mathcal{T}}$ is induced by a discrete cone, and if we are able to identify the set $\mathcal{U}_{\mathcal{T}}$ such that the following statements are equivalent:

(a) $\forall m, m' \in \mathbb{Z}^{|\mathcal{S}|}: \quad m \succeq_{\mathcal{T}} m' \implies W_u(m) \geq W_u(m')$,

(b) $u \in \mathcal{U}_{\mathcal{T}}$,

Then we know from the previous theorem that the following statements are also equivalent:

(a) $m \succeq_{\mathcal{T}} m'$,

(b) $W_u(m) \geq W_u(m'), \forall\, u \in \mathcal{U}_{\mathcal{T}}$.

This result is useful to simplify the proof of most of the HLP-type theorems in the literature.

3.2. Outline of the Proof

(a)	$\iff$	(b)	$\iff$	(c)
$m \succeq_{\mathcal{T}} m'$		$W(m) \geq W(m'), \forall\, W \in \mathcal{W}_{\mathcal{T}}$		$W_u(m) \geq W_u(m'), \forall\, u \in \mathcal{U}_{\mathcal{T}}$

(**a**) $\Longrightarrow$ (**b**). That results from the definition of $\mathcal{W}_{\mathcal{T}}$.

(**b**) $\Longrightarrow$ (**a**). For all $z \in \mathbb{Z}^{|\mathcal{S}|}$, choose $W_m(z) = 0$ if $(m - z) \in \mathcal{D}(\mathcal{T})$, and $W_m(z) = 1$ otherwise. One observes that $W_m \in \mathcal{W}_{\mathcal{T}}$. Moreover $(m - m) = 0 \in \mathcal{D}(\mathcal{T})$, hence $W_m(m) = 0$. If (b) is true, then $W_m(m') = 0$ and thus $(m - m') \in \mathcal{D}(\mathcal{T})$.

(**a**) $\Longrightarrow$ (**c**). Follow directly from the proof $(b) \Rightarrow (a)$ in Theorem 1.

(**c**) $\Longrightarrow$ (**a**). **1)** Let $\mathcal{U}_{\mathcal{T}}^{\circ}$ and $\mathcal{T}^{\circ}$ be the *polar cones* of $\mathcal{U}_{\mathcal{T}}$ and $\mathcal{T}$. If (a) is true, then $(m - m') \in \mathcal{U}_{\mathcal{T}}^{\circ}$. By definition $\mathcal{U}_{\mathcal{T}} = \mathcal{T}^{\circ}$, hence $(m - m') \in \mathcal{T}^{\circ\circ}$. Precisely $(m - m') \in \mathcal{T}^{\circ\circ} \cap \mathbb{Z}^{|\mathcal{S}|}$.

2) By the *bipolar theorem*, $\mathcal{T}^{\circ\circ} = \mathrm{co}\{\lambda m \,|\, \lambda \in \mathbb{R}_+ , m \in \mathcal{T}\}$. Because $\mathcal{T}^{\circ\circ}$ is also a *rational cone*, it is generated by a *minimal Hilbert basis* (which appears to be $\mathcal{T}$), such that any element in $\mathcal{T}^{\circ\circ} \cap \mathbb{Z}^{|\mathcal{S}|}$ is in $\mathcal{D}(\mathcal{T})$.

Applications

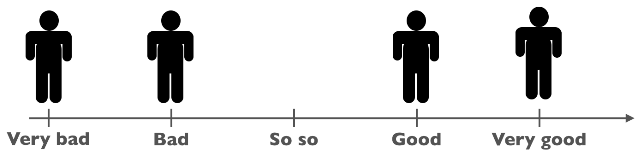
4.1. Ordinal variables and Hammond transfers

A discrete framework is relevant for most of the *cardinal variables* used in practice, even those usually treated as continuous variables.

Example: income, defined in euro cents.

But a discrete framework is required for *ordinal variables* defined on a discrete scale, namely *ordered categorical variables*.

Example: Self-reported health

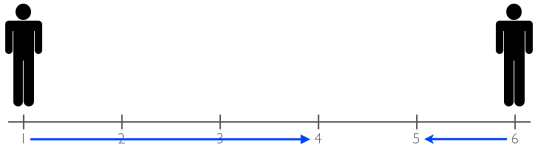


In that case, a numerical representation of the scale is allowed, but defined up to an increasing transformation.

5.1. Ordinal variables and Hammond transfers (2)

For an ordered categorical variable, Gravel et al. (2014) propose the notion of *Hammond transfer* (in reference to Hammond, 1976).

This is equivalent to a Pigou-Dalton transfer (from a 'rich' to a 'poor') but without the mean-preserving condition (senseless here):



Here the set of outcomes is simply $\mathcal{S} = \{1, 2, \ldots, K\}$, a fixed list of ordered categories.

A distribution is the nb. of ind. in each category, $n = (n_1, n_2, \ldots, n_K)$.

5.1. Ordinal variables and Hammond transfers (3)

We denote by $\mathcal{T}_H$ the set of Hammond transfers, and by $\succeq_H$ the induced discrete cone. Let:

$$\mathcal{U}_H = \left\{ u \in \mathbb{R}^K \mid (u_j - u_i) \geq (u_l - u_k) \text{, for all } 1 \leq i < j \leq k < l \leq K \right\}$$

Gravel et al. (2014) have shown the following equivalence:

(a) $\forall n, n \in \mathcal{N} : \quad n \succeq_{\mathcal{T}_H} n' \implies W_u(n) \geq W_u(n')$,

(b) $u \in \mathcal{U}_H$.

They have also established the relationships $(a) \Rightarrow (b) \Leftrightarrow (c)$ below:

(a) $n \succeq_{\mathcal{T}_H} n'$,

(b) $W_u(n) \geq W_u(n')$, for all $u \in \mathcal{U}_H$,

(c) $H(k; n) \leq H(k; n')$ and $\bar{H}(k, n) \leq \bar{H}(k, n')$ for all $k \in \mathcal{S}$.

Because $\succeq_{\mathcal{T}_H}$ *is induced by a discrete cone*, from Theorem 2 we also have $(b) \Rightarrow (a)$. The equivalence theorem is completed!

4.2. Transfers not-consistent with the framework

The transfers we consider, defined as the difference between two distributions, satisfy the following property, somewhat restrictive.

Remark (Independence)

Let $n, n', m, m' \in \mathcal{N}$ such that $m = n + \epsilon$ and $m' = n' + \epsilon$. If n is obtained from n' by a transfer in $\mathcal{T}$, then m is also obtained from m' by a transfer in $\mathcal{T}$.

This result is immediate : if $(n - n') \in \mathcal{T}$, then $(m - m') \in \mathcal{T}$. Hence some transfers in the literature are excluded.

For income distributions, Chateauneuf and Moyes (2006) distinguish:

$\rightarrow$ *Pigou-Dalton transfers*: mean-preserving progr. transfers (PD),

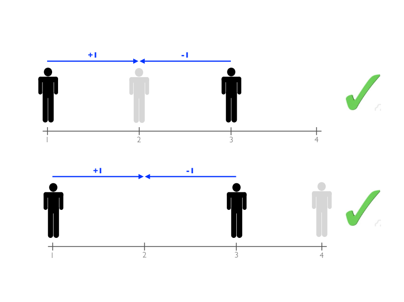
$\rightarrow$ *Uniform PD transfers*: 'solidarity' among the rich and poor (UPD).

4.2. Transfers not-consistent with the framework (2)

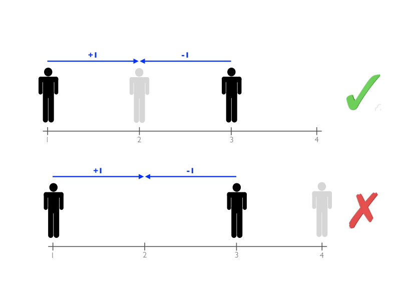
PD transfers satisfies *Independence*, unlike UPD transfers.

Example: Consider the income scale $\mathcal{S} = \{1, 2, 3, 4\}$, the distribution $n = (1, 1, 1, 0)$ and the vector $\epsilon = (0, -1, 0, 1)$. $m = n + \epsilon = (1, 0, 1, 1)$.

PD



UPD



The transfer is this example is a PD transfer for n and m, a UPD transfer for n but not an UPD transfer for m.

4.2. Transfers not-consistent with the framework (3)

Chateauneuf and Moyes (2006) have shown that, for $\theta = \mathcal{T}_{PD}, \mathcal{T}_{UPD}$, the following statements are equivalent:

(a) For all $x, x' \in \mathbb{R}^d$ with equal means: $x \succeq_\theta x' \implies W_u(x) \geq W_u(x')$,

(b) u is *concave*.

Moreover, for equal mean distributions, it is well-known that the following statements are also equivalent:

(a) $x \succeq_{\mathcal{T}_{PD}} x'$,

(b) $W_u(x) \geq W_u(x')$, for all u concave.

Because $\succeq_{\mathcal{T}_{UPD}} \subset \succeq_{\mathcal{T}_{PD}}$, it is not true that (b) is equivalent to $x \succeq_{\mathcal{T}_{PD}} x'$.

The two results cannot be linked for $\succeq_{\mathcal{T}_{UPD}}$: *This is not a cone ordering*.

Conclusion

We have propose a abstract model which can help at completing HLP theorem, when the variables under consideration are discrete.

The only requirement is that the considered set of transfers can be described as a discrete cone, which is usually the case.

The discrete framework offers perspectives, in social welfare measurement but also in the theory of decision under risk.

It appears that the expected utility model (utilitarianism) plays a central role: this is a ‘basis’ for any social preferences model.

Extension: this work is focus on the ‘first part’ of the HLP theorems.
Is this approach helpful to identify implementation criteria?

Thank you for your attention

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