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A theory of multidimensional equality of opportunity

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The plan of the presentation

- EOp in the unidimensional case
- Motivation & what we do
- The analytical framework
- Class (1, 2, 3) SEFs:
  - Axioms
  - Dominance results
  - IOp measures
- Conclusions

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Equality of opportunity in the unidimensional case

- The canonical model of Equality of Opportunity (Roemer, 1998; Fleurbaey, 2008)
- See Francisco Ferreira’s lecture tomorrow

Equality of opportunity in the unidimensional case

- From {outcomes, circumstances, effort} to opportunities:

  $x = g(c, e)$

- x outcome, e responsibility, c circumstances

- Compensate for the circumstances (**compensation principle**)
  - Ex ante: equal opportunity sets (*before effort is chosen)*
  - Ex post: equal outcome for equal effort

- Reward the effort (**reward principle**)
  - Utilitarian reward
  - Agnostic reward
  - Inequality averse reward
  - Liberal, minimal, …

Equality of opportunity in the unidimensional case

Consider the matrix

$$X = \begin{bmatrix} & \mathbf{e}_1 & \dots & \mathbf{e}_j & \dots & \mathbf{e}_m \\ \mathbf{c}_1 & x_{1,1} & \dots & x_{1,j} & \dots & x_{1,m} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \mathbf{c}_i & x_{i,1} & \dots & x_{i,j} & \dots & x_{i,m} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \mathbf{c}_n & x_{n,1} & \dots & x_{n,j} & \dots & x_{n,m} \end{bmatrix} \quad \text{with associated } p_{ij}$$

- Each row is a type distribution, X_i , interpreted as opportunity set
- Each column is a tranche distribution, X^j.
- Ex ante approach: Focus on inequality between types
- Ex post approach: Focus on inequality within tranches

Measuring inequality of opportunity in the unidimensional case

- **Testing for EOp**

  Lefranc et al. (2008)

- **Partial orderings: dominance conditions**
  - Andreoli et al. (2019), Peragine (2002, 2004)

- **Complete orderings: IOp measures**
  - Almas et al. (2011), Bourguignon et al. (2007), Checchi and Peragine (2010), Ferreira and Gignoux (2011)

  - www.equalchances.org

Measuring inequality of opportunity in the unidimensional case

- Peragine (2004)
  - Uses an axiomatic approach
  - Identifies classes of Social welfare functions
  - Obtain dominance conditions

- Ex ante axioms:
  - Monotonicity
  - Additivity within and between types
  - Ex ante Compensation: Inequality aversion between types
  - Reward:
    - Utilitarian reward: Inequality neutrality within types
    - Agnosticism and inequality aversion

- See also Bosmans and Ozturk (2017), Fleurbaey et al. (2018), Brunori et al. (2015)

Measuring inequality of opportunity in the unidimensional case

- Peragine (2004) in the ex ante approach by using (Monotonicity, Additivity, Symmetry within types, Inequality neutrality within types, Inequality aversion between types) obtains the following dominance conditions: *F > G* iff

$$\sum_{i=1}^{k} q_i^F \mu_i^F \geq \sum_{i=1}^{k} q_i^G \mu_i^G, \forall k \in \{1, ..., n\}$$

- evaluate opportunity set by (weighted) mean: $q_i^F \mu_i^F$

- compare distributions of opportunity sets $\left(q_1^F \mu_1^F, ..., q_n^F \mu_n^F\right)$ by generalized Lorenz dominance

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Motivation & what we do

- We extend EOP theory to the case in which individual **outcome is a multidimensional variable** (e.g. income, health, education).
- We focus on **ex-ante** approach.
- We have a multidimensional distribution of outcomes within each type.
- Types differ in marginal distributions and dependence. **What is the extent of unfair opportunities if one takes both into account?**
- How does considering joint distribution of outcomes affect measured IOP?

Motivation & what we do

- Our goal is to order distributions of multidimensional outcomes X by a preference of an "opportunity egalitarian social decision maker" represented by a continuous social welfare function.
- We focus on three classes of social welfare functions, characterized by certain EOP axioms.
- All the classes satisfy **ex ante compensation.**
- The first class satisfies **utilitarian reward** (i.e., it is neutral to inequality within type)
- The second class satisfies **inequality-averse reward principle** (i.e., it is averse to inequality within type).
- The third class is **agnostic with respect to reward.**
- We adopt a dual perspective: welfare functions and inequality measures, which are **induced** by welfare functions.

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The analytical framework

- We have a society consisting of N individuals described by a vector of circumstances $O \in \mathcal{O}$ (ordered $O_i \prec O_{i+1}$) and effort (scalar) $w \in \Theta \subseteq \mathbb{R}_+$.
- Outcome is generated deterministically by $g : \mathcal{O} \times \Theta \to \mathbb{R}^k_+$.
- Let $D = \{X \in M_{N \times k}(\mathbb{R}_+) : g \text{ is monotone in } \mathcal{O}, \Theta\}$ denote the set of possible outcome profiles.
- $W : D \to \mathbb{R}$ is a continuous welfare function and $I : D \to [0, 1]$ is an inequality index.
- Notation: X^h_{ij} — i-th individual, j-th dimension, h type. X^h a distribution within a type h. X_μ rows are type-means on each dimension (equality *within* type but not *between* type). X^μ a matrix of population means (perfect equality).

IOp measures

- Inequality measure I_W is induced by a welfare function W if $I_W(X) = 1 - \delta(X)$ where $\delta(X) \in [0, 1]$ satisfies equation $W(X) = W(\delta(X)X^{\mu})$, where X^{μ} is a matrix of means (i.e. perfect equality within and between types).
- Then indices are normatively significant i.e. *under some restrictions e.g. equality of means* $W(X) \leq W(Y) \iff I(X) \geq I(Y)$.
- We care about inequality because we believe that it lowers welfare (Dalton 1920, Atkinson 1970).

Definitions

- A Pigou-Dalton Transfer in the multivariate context is a transfer between two individuals that simultaneously involves all attributes (but we admit different proportions).

PDT — between individuals i_1, i_2 from type h we make a transfer on each dimension j of potentially different amounts $X^h_{i_1j} = Y^h_{i_1j}\varepsilon_j + Y^h_{i_2j}(1-\varepsilon_j)$ and $X^h_{i_2j} = Y^h_{i_2j}(1-\varepsilon_j) + Y^h_{i_1j}\varepsilon_j$ where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ with $\varepsilon_i \geq 0$ and at least one $\varepsilon_i > 0$.

Definitions

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- A correlation-increasing transfer is an exchange of all attributes between two individuals after which one individual is left with the lowest endowment and the other with the maximum endowment of each attribute. Such an operation clearly increases correlation between dimensions.

CIT — correlation - increasing transfer (Tsui 1999). Individual i_1 is given $\left(\max X^h_{i_1 j}, X^h_{i_2 j}\right)$ and individual i_2 gets $\left(\min X^h_{i_1 j}, X^h_{i_2 j}\right)$ on each dimension j

Definitions

- All attributes are assumed to be transferable
- Does it make much sense to talk of “transferring health”?
- Bosmans et al. (2009) study the implications of formulating a version of the Pigou-Dalton principle that applies only to transferable attributes
- Muller and Trannoy (2012) examine dominance conditions when attributes are asymmetric in the sense that one attribute (typically income) can be used to compensate for lower levels of other attribute(s) (e.g. needs, health, etc.).

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Class 1: Axioms

- **MONOTONICITY (MON)** $W : D \to \mathbb{R}$ is a monotone function
- **ADDITIVITY (ADD)** $W(X) = \sum_{h=1}^{n} \sum_{i=1}^{N_h^X} U^h(X_i^h)$ (*can be relaxed to adding utilities within type, but general aggregation between types)

Class 1: Axioms

Utilitarian Reward:

- **TYPE SYMMETRY (T-SYM)** W is invariant to permutation of individuals *within a type*
- **INEQUALITY NEUTRALITY WITHIN TYPES (INWT)** For all $X, Y \in D$, if $X^h = Y^h$ for all $h \neq l$, X^l is obtained from Y^l by PDT or CIT, then $W(X) = W(Y)$.

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Ex ante Compensation:

- **INEQUALITY AVERSION BETWEEN TYPES (IABT)** For all $X, Y \in D$, X is obtained from Y via a PDT between two types and the ordering of types $\mathcal{O}$ is unchanged. Then $W(X) > W(Y)$.

Class 1: Definitions

- Altogether, Class 1 is

$$\mathcal{W}^{AOEN} = \{W | \mathrm{MON}, \mathrm{ADD}, \mathrm{T-SYM}, \mathrm{INWT}, \mathrm{IABT}\}.$$

Class 1: Results

- **THEOREM 1** For $X, Y \in D$ we have

$$X_\mu \succeq_{CGLD} Y_\mu \iff W(X) \geq W(Y) \quad \forall_{W \in \mathcal{W}^{AOEN}},$$

  where CGLD is Generalized Lorenz Dominance applied to *each* dimension *separately* (C – component-wise).
- Because of neutrality, type distribution X^h can be summarized by type-means distribution X_μ.
- Necessary for a meaningful interpretation of CGLD: total sum on *each* dimension has to be higher in one type.
- A restrictive result - as a consequence of INWT utility functions are affine and W is of the form

$$W(X) = \sum_{h=1}^{n} \sum_{i=1}^{N_h^X} \sum_{j=1}^{k} a_j^h X_{ij}^h,$$

  with $a_j^h > a_j^{h+1}$ for all j.

Class 1: Results

- Component-Wise Lorenz Dominance is non-welfarist, in the sense that the evaluation of respective distributions depends directly on the values of the attributes.

Class 1: Extensions

SEPARABILITY BETWEEN TYPES (SBT) There exist function $\psi : \mathbb{R}^n \to \mathbb{R}$, and for all $h = 1, \ldots, n$ there exist functions $u_h : \mathbb{R}^{N_h^X} \to \mathbb{R}$ and $U^h : \mathbb{R}^k \to \mathbb{R}$ assumed to be twice differentiable (almost everywhere), such that $u_h = \sum_{i=1}^{N_h^X} U^h(X_i^h)$ and $W(X) = \psi(u_1, \ldots, u_n)$.

INEQUALITY AVERSION BETWEEN TYPES (IABT*) For all $X \in D$, $\epsilon > 0$ we have

$$\psi(u_1, \ldots, u_n) < \psi(u_1, \ldots, u_p + \epsilon, \ldots, u_q - \epsilon, \ldots, u_n)$$

where $p < q$ and ε such that ordering of $\mathcal{O}$ is unchanged.

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Class 2: Axioms and Definitions

Inequality Averse Reward

- **INEQUALITY AVERSION WITHIN TYPES (IAWT)** For all $X, Y \in D$, if $X^h = Y^h$ for all $h \neq l$, X^l is obtained from Y^l by PDT or CIT, then $W(X) < W(Y)$.
- We keep **IABT, MON, ADD, T-SYM**
- Altogether, Class 2 is

$$\mathcal{W}^{AOEA} = \{W | \mathrm{MON}, \mathrm{ADD}, \mathrm{T-SYM}, \mathrm{IAWT}, \mathrm{IABT}\}.$$

Class 2: results

- **THEOREM 3** For $X, Y \in D$ we have

$$X \succeq_{LD(\mathcal{U}^{ICL})} Y \iff W(X) \geq W(Y) \quad \forall_{W \in \mathcal{W}^{AEOA}},$$

where

$$X \succeq_{LD(\mathcal{ICL})} Y \iff \sum_{h=1}^{l} u_h^X \geq \sum_{h=1}^{l} u_h^Y \quad \forall_{l=1,\ldots,n} \forall_{U^h \in \mathrm{ICL}},$$

and

$$\mathcal{U}^{ICL} = \{U | \mathrm{Increasing}, \mathrm{Type} - \mathrm{Concave}, \mathrm{Submodular}\}.$$

Definition 4. ***Type-Concavity*** *Function $U^h : \mathbb{R}^k \to \mathbb{R}$ is type-concave if its first derivate decrease with respect to a type i.e. the better the type the lower the first derivative. Formally, $dU^h/dX > dU^{h+1}/dX > 0$.*

Definition 5. ***Submodularity*** *Function U^h is submodular, if $U^h(X_p^h)+U^h(X_q^h) > U^h(X_p^h \wedge X_q^h)+U^h(X_p^h \vee X_q^h)$ where $X_p^h \wedge X_q^h$ is a vector of elements $\max\{X_{pj}^h, X_{qj}^h\}$ and $X_p^h \wedge X_q^h$ of $\min\{X_{pj}^h, X_{qj}^h\}$*

Submodularity reflects that association between dimensions matters, and if there is more of it the utility is lower.

Class 2: Results

- LD first aggregates individual utilities within type (note that an individual utility is a function of many attributes), thereby obtaining a value of type opportunity set; and then compares partial sums of such aggregate utility vectors.

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Class 3

Agnostic with respect to reward

$$\mathcal{W}^{AOEAG} = \{W | \mathrm{MON}, \mathrm{ADD}, \mathrm{T-SYM}, \mathrm{IABT}\}$$

$$\mathcal{U}^{ICLA} = \{U | \mathrm{Increasing}, \mathrm{Type-Concave}\}.$$

Theorem 3. $X \succeq_{LD(\mathcal{U}^{ICAL})} Y \iff W(X) \geq W(Y) \quad \forall_{W \in \mathcal{W}^{AEOAG}}$

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IOp measures

Definition

I is inequality measure if it satisfies the following properties

1. I is continuous,
2. $I(X^{\mu}) = 0$,
3. $I(X) = I(\sigma(X))$ for any permutation σ satisfying T-SYM i.e. *within* types only,
4. $I(Y) < I(X)$ if Y is PDT *between types* of X <- aversion to inequality *between* types.

Measure is ***relative*** if additionally $I(XC) = I(X)$ for diagonal matrix C. If $I(XC) = I(X)$ for C diagonal with equal elements, then I is ***weakly relative***.

Class 1 IOp Measures: Results

THEOREM 2

1. I_W is given by $1 - \frac{W(X_\mu)}{W(X^\mu)} = 1 - \frac{\sum_{h=1}^{n} N_h^X \sum_{j=1}^{k} a_j^h (X_\mu)_{1j}^h}{\sum_{h=1}^{n} N_h^X \sum_{j=1}^{k} a_j^h (X^\mu)_{1j}^h}$,
2. Here $a_j^h > a_j^{h+1}$: better off type gets lower weight
3. I_W is a weakly relative inequality measure.

- The index related to the class 1 is one minus the weighted sum of type-means for each dimension normalized by the highest amount of welfare achievable

- It is a weakly relative measure: it does not change when all attributes are scaled by the same factor, but it is not invariant when each attribute is scaled by its mean.

Class 2 IOp measures

- Further restriction on the Class 2
  RATIO SCALE INVARIANCE (RSI) (Tsui 1995)
  $W(X) = W(Y) \iff W(XC) = W(YC)$ for C diagonal

Class 2 IOp Measures

THEOREM 4

1. $I_W(X)$, is a relative inequality measure,
2. Utility functions U^h are of the form $a_h \prod_{j=1}^{k} (X_{ij}^h)^{r_j}$, $r_j \in (0, 1]$,
3. $I_W(X)$ is given by

$$I_W(X) = 1 - \left(\sum_{h=1}^{n} w_h \frac{U^h((X_\mu)_1^h)}{U^h((X^\mu)_1^h)} \right)^{\frac{1}{\sum_{j=1}^{k} r_j}}$$

where $w_h = \frac{\delta_h(X) N_h^X}{N}$ for $\delta_h(X)$ of form

$$\delta_h(X) = \left[\frac{1}{N_h^X} \sum_{i=1}^{N_h^X} \prod_{j=1}^{k} \left(\frac{X_{ij}^h}{(X_\mu)_{1j}^h} \right)^{r_j} \right].$$

Class 2 measures

- Inequality indices related to Class 2 are weighted sums of normalized types' utilities, where weights are Tsui (1995) inequality indices computed within type.

- Two components: the distribution of utilities *between* types $\left(\frac{U^h\left((X_\mu)_1^h\right)}{U^h\left((X^\mu)_1^h\right)}\right)$ and the distribution of attributes *within* type $(\delta_h(X)$.
- Weights w_h are Tsui (1995) inequality indices computed *within* a type – sensitivity to attributes' dependence.
- Due to concavity of I_W a more equal distribution of $U^h((X_\mu)_1^h)$ is preferred – inequality aversion between types' welfare.
- For $r_j = r$ for all j, an increase in r results in a decrease of inequality. r_j are dimensions' weights.

- Parameters r are dimensions weights. The higher r the higher the degree of concavity in a given dimension and the higher inequality weight attached to this dimension.

Conclusions

- We have incorporated multidimensionality of outcomes in the canonical model of EOp, and characterized dominance conditions which pays attention to it.

- We also characterized classes of “induced” IOP measures

- Further research:
  - (i) domain extensions (type partitions)
  - (ii) allowing for transferable and non transferable attributes
  - (iii) different degrees of inequality aversion for circumstances-based and effort-based inequalities
  - (iv) extension of the ex post approach, in all its variants (see Roemer, 1998, Fleurbaey 2008 and Fleurbaey et al. 2017);
  - (v) empirical analysis.

Thank you.