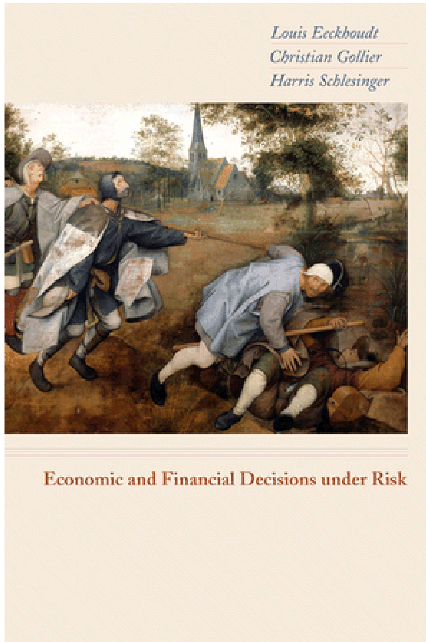
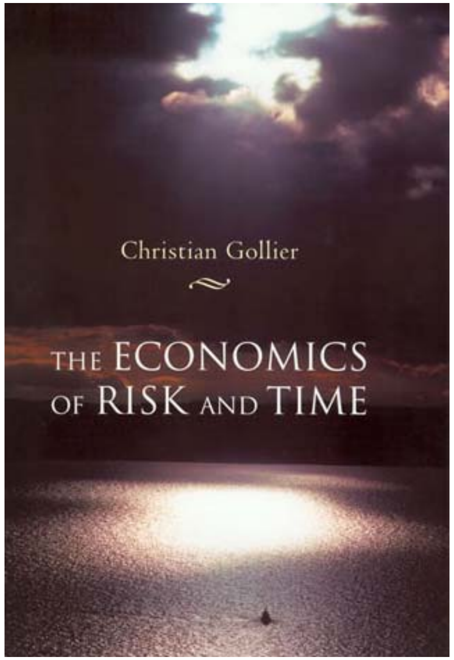
A simple mathematical problem

The cake-sharing problem



Chapter 11



Chapter 21

Definition of the problem

- Let $\theta = 1,\ldots,N$ an index for individuals;
- $u_\theta : \mathrm{R} \to \mathrm{R}$: an increasing and concave function;
- $(\lambda_1,\ldots,\lambda_N)$: a vector of positive Pareto weights.
- $(h_1,\ldots,h_N)$: a vector of positive proportions.

$$v(z) = \max_{(c_1,\ldots,c_N)} \sum_{\theta=1}^{N} \lambda_\theta h_\theta u_\theta(c_\theta)$$

$$s.t. \sum_{\theta=1}^{N} h_\theta c_\theta = z.$$

Property 1

$$v(z) = \max_{(c_1,...,c_N)} \sum_{\theta=1}^{N} \lambda_\theta h_\theta u_\theta(c_\theta)$$

$$s.t. \sum_{\theta=1}^{N} h_\theta c_\theta = z.$$

- For a given z, this problem has a single solution, which satisfies the following conditions:

  $\lambda_\theta u_\theta'(c_\theta(z)) = \xi(z)$ for all θ.

- Suppose that $u_\theta = u$ for all θ.

  $(c_1(z),...,c_N(z))$ and $(\lambda_1,...,\lambda_N)$ are comonotone.

$$v(z) = \max_{(c_1,\ldots,c_N)} \sum_{\theta=1}^{N} \lambda_\theta h_\theta u_\theta(c_\theta)$$

$$s.t. \ \sum_{\theta=1}^{N} h_\theta c_\theta = z.$$

Property 2

- $\lambda_\theta u_\theta{}'(c_\theta(z)) = \xi(z)$ for all θ.

$$\Rightarrow \lambda_\theta u_\theta{}''(c_\theta(z)) c_\theta{}'(z) = \xi'(z) \text{ for all } \theta.$$

$$\Rightarrow c_\theta{}'(z) = -\frac{\xi'(z)}{\xi(z)} T_\theta(c_\theta(z)) \text{ for all } \theta.$$

$$\Rightarrow 1 = \sum_{\theta=1}^{N} h_\theta c_\theta{}'(z) = -\frac{\xi'(z)}{\xi(z)} \sum_{\theta=1}^{N} h_\theta T_\theta(c_\theta(z))$$

$$\Rightarrow \boxed{c_\theta{}'(z) = \frac{T_\theta(c_\theta(z))}{\sum_{t=1}^{N} h_t T_t(c_t(z))}}$$

Property 3

- Analysis of the SWF $v(z) = \sum_{\theta=1}^{N} \lambda_\theta h_\theta u_\theta(c_\theta(z))$.

$$v'(z) = \sum_{\theta=1}^{N} \lambda_\theta h_\theta u'_\theta(c_\theta(z)) c'_\theta(z) = \xi(z) \sum_{\theta=1}^{N} h_\theta c'_\theta(z) = \xi(z).$$

$$v''(z) = \xi'(z).$$

$$T_v(z) = -\frac{v'(z)}{v''(z)} = -\frac{\xi(z)}{\xi'(z)} = \sum_{\theta=1}^{N} h_\theta T_\theta(c_\theta(z)).$$

$$T_v(z) = \sum_{\theta=1}^{N} h_\theta T_\theta(c_\theta(z)).$$

A special case

- Suppose ISHARA: $T_\theta(c) = t_\theta + bc \quad \textit{for all } \theta.$

$$c_\theta{}'(z) = \frac{t_\theta}{\sum_{\tau=1}^{N} t_\tau} \quad \text{(or an arbitrary constant when } t_\theta = 0 \; \forall \theta)$$

$$T_v(z) = \sum_{\theta=1}^{N} t_\theta + bz.$$

- T_v is independent of *(λ_1,...,λ_N)*.

Some exotic applications

(exotic wrt Canazei seminar participants)

Saving and lifetime utility

$$v(z) = \max_{(c_1,\ldots,c_N)} \sum_{\tau=1}^{N} \beta^{\tau} u_{\tau}(c_{\tau})$$

$$s.t. \sum_{\tau=1}^{N} \frac{c_{\tau}}{(1+r)^{\tau}} = z.$$

- Application: How does the ability to reallocate risk over time affect risk taking?

CARA case:

$$T_v(z) = \sum_{\theta=1}^{N} t_{\theta} = Nt$$

Arrow-Debreu portfolio choice

$$v(z) = \max_{(c_1,\dots,c_S)} \sum_{s=1}^{S} p_s u_s(c_s)$$

$$s.t. \ \sum_{s=1}^{S} \Pi_s c_s = z.$$

- Application: Should younger people take more portfolio risk ?

CRRA case: $T_v(z) = bz = T(z)$

Efficient risk-sharing and efficient collective risk-taking

Aggregation of heterogeneous risk attitudes

The collective choice problem

- A group of N risk-averse VNM agents.
- S possible states of nature with prob $(p_1,...,p_S)$.
- Endowment of the cake per capita in state s: z_s.
- The group shares risk efficiently according to $(\lambda_1,...,\lambda_N)$, which is exogenous.
- The group can insure risks on Arrow-Debreu markets with prices $(\pi_1,...,\pi_S)$.

The collective choice problem

$$\max_C \sum_{\theta=1}^{N} \lambda_\theta \sum_{s=1}^{S} p_s u_\theta(c_{\theta s})$$

$$s.t. \quad \sum_{s=1}^{S} \pi_s \left(\left(\frac{1}{N} \sum_{\theta=1}^{N} c_{\theta s} \right) - z_s \right) = 0.$$

Additivity of (1) the SWF and (2) EU.

$$\max_C \sum_{s=1}^{S} p_s \sum_{s=1}^{S} \lambda_\theta u_\theta(c_{\theta s})$$

$$s.t. \quad \sum_{s=1}^{S} \pi_s \left(\frac{1}{N} \left(\sum_{\theta=1}^{N} c_{\theta s} \right) - z_s \right) = 0.$$

The representative agent

$$\max_C \sum_{s=1}^{S} p_s \sum_{s=1}^{S} \lambda_\theta u_\theta (c_{\theta s})$$

$$s.t. \quad \sum_{s=1}^{S} \pi_s \left(\left(\sum_{\theta=1}^{N} c_{\theta s} \right) - w_s \right) = 0.$$

$$v(z) = \max_{(c_1,\ldots,c_N)} \sum_{\theta=1}^{N} \lambda_\theta u_\theta (c_\theta)$$

$$s.t. \quad \frac{1}{N} \sum_{\theta=1}^{N} c_\theta = z.$$

The cake sharing problem

$$\max_C \sum_{s=1}^{S} p_s v(z_s)$$

$$s.t. \quad \sum_{s=1}^{S} \pi_s \left(z_s - w_s \right) = 0.$$

The collective risk exposure problem

The mutuality principle: Elimination of diversifiable risks

- Proposition: If there are two states *s* and *s*' such that $z_s = z_{s'}$, then $c_{\theta s} = c_{\theta s'}$ for all θ.
- This eliminates all diversifiable risks in the group.
- This means that $c_{\theta s} = C_\theta(z_s)$ with

$$C_\theta'(z) = \frac{T_\theta(\mathrm{C}_\theta(z))}{\frac{1}{N}\sum_{t=1}^{N} T_t(\mathrm{C}_t(z))}$$

Sharing the aggregate risk

$$C_\theta'(z) = \frac{T_\theta(C_\theta(z))}{\frac{1}{N}\sum_{t=1}^{N} T_t(C_t(z))}$$

Measure the risk borne by agent θ, locally.

⇩

Those with a smaller risk tolerance bear a smaller share of the risk.

Two-fund separation theorem

- If $C_\theta(z)$ is linear in z, agent θ's portfolio is a combination of the risk free asset and the market portfolio.
- Two funds: the risk free fund and the market portfolio. (Cass and Stiglitz (1970))
- Proposition: *All agents select the same portfolio of risky assets (the market portfolio) if* $T_\theta(c)=t_\theta+bc$ *(ISHARA).*

The case of small risk

- Suppose that the GDP per capita is initially certain (z).
- It generates an allocation $(C_1(z),...,C_N(z))$.
- Now, there is a small aggregate risk Y per capita to be shared, with $EY=0$.
- Let $a(\theta)$ be the share of the risk borne by agent θ.
- $CE_\theta=C_\theta(z)-0.5a_\theta^2\sigma^2A_\theta$

The case of small risk

The optimal sharing rule maximizes the sum of the certainty equivalent consumptions:

$$CE = \max_{a(.)} \sum_{\theta=1}^{N} CE_{\theta} \quad \text{s.t.} \sum_{\theta=1}^{N} \mathrm{a}_{\theta} = 1$$

$$FOC : \sigma^2 a_{\theta} A_{\theta} = \psi$$

$$\Rightarrow a_{\theta} = \frac{T_{\theta}}{\sum_{\tau} T_{\tau}}.$$

$$\Rightarrow CE = z - 0.5 \sum_{\theta} a_{\theta}^2 \sigma^2 A_{\theta} = z - \frac{0.5\sigma^2}{\sum_{\theta} T_{\theta}}$$

Group decision process

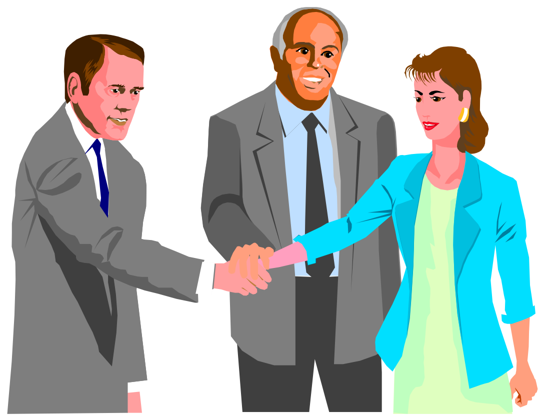
The group is offered to take risk Y. How does the group take the decision to accept or to reject this risk?

Using the optimal sharing rule :

$$CE = z + EY - 0.5\sigma^2 \left[T_v\right]^{-1}.$$

$$CE_\theta = C_\theta(z) + \frac{T_\theta}{T_v}\left[EY - 0.5\sigma^2\left[T_v\right]^{-1}\right].$$

Unanimity!



More on unanimity

- All agents have the same attitude towards the aggregate risk if $v_\theta(z) = u_\theta(C_\theta(z))$ has the same degree of concavity with respect to z for all θ.

$$-\frac{v_\theta''(z)}{v_\theta'(z)} = \frac{1 + T_v'(z)}{T_v(z)} - \frac{T_\theta'(C_\theta(z))}{T_v(z)}$$

- Proposition: *There is unanimity if and only if the group has the ISHARA property.*

The representative agent

- The group behaves toward the risk per capita as a single expected-utility maximizing person with utility function *v*.
- This is in spite of the fact that the optimal risk-sharing behind *v* can be very complex.

$$v(z) = \max_{(c_1,\ldots,c_N)} \sum_{\theta=1}^{N} \lambda_\theta u_\theta(c_\theta)$$

$$s.t. \quad \frac{1}{N}\sum_{\theta=1}^{N} c_\theta = z.$$

The representative agent

- Assuming that all agents are identical is not restrictive when markets are complete.
- What is more difficult is to assess the degree of risk aversion of the representative agent in an heterogeneous economy.

The effect of wealth inequality

- Suppose that all agents have the same utility function.
- We examine the impcat of wealth inequality on the collective risk tolerance.
- Egalitarian economy: $C_\theta(z)=z$.
- Unequal economy: The Pareto weights are heterogeneous.
- Assume that T is concave.

$$T_v(z) = \frac{1}{N}\sum_\theta T(C_\theta(z)) \leq T\left(\frac{1}{N}C_\theta(z)\right) = T(z).$$

The effect of wealth inequality

- Proposition: Wealth inequality decreases the collective degree of risk tolerance if and only if T is concave.
- Gollier (2001).
- ISHARA: no effect of wealth inequality on the collective risk attitude.

Who should we believe?

Collective risk-taking decisions with heterogeneous beliefs

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Motivation

- People agree to disagree on the likelihood of
  - global warming, bad effect of GMOs, ...;
  - the Big One in the L.A. area next year;
  - an economic boom in Europe next year;
  - the success of a new technology;...
- No asymmetric information.

Collective choices

- Given these divergent opinions,
  - how are risks priced by the market?
  - should we, as a group,
    - reduce gas emissions; prohibit GMOs,…;
    - reinforce earthquake-resistance building standards?;
    - invest more in the new technology?
- What probability distribution should we use in collective decision-making?

Relaxing the « Harsanyi doctrine »

- Harsanyi doctrine: all agents share common prior beliefs.
- Why this may not be the case?
  - Non Expected Utility:
    - people distort probabilities;
    - heterogeneous degrees of ambiguity aversion;
  - Economics and psychology:
    - negative value of information in the absence of commitment device and hyperbolic discounting;
    - anticipatory feelings and preference-induced optimal beliefs.

Assumption: efficient risk sharing

- Our central assumption is that risks are shared in a Pareto-efficient way.
- For example: complete markets for Arrow-Debreu securities.
- Alt: social security, implicit insurance,…
- The more risk-averse agents will be insured by the more risk-tolerant ones.
- The more pessimistic agents will be insured by the more optimistic ones.

Aggregation of beliefs in an efficient group

- A simple idea: only those members of the group who bear a share of the risk will see their beliefs taken into account in the collective risk perception.
- A simple result: if agent θ's risk tolerance equals $k\%$ of the group's risk tolerance,
  - he will bear $k\%$ of the group's risk;
  - his beliefs will count for $k\%$ of the group's beliefs.
- Local property.

Related literature

- Aggregation problem with *homogenous* beliefs: Borch (1960), Constantinides (1982), Hara and Kuzmics (forthcoming, JET),…

- Aggregation problem with *heterogeneous* beliefs: Wilson (1968), Rubinstein (1974), Leland (1980), Varian (1985), Ingersoll (1987), Calvet, Grandmont and Lemaire (2002), Jouini and Napp (2003).

Structure of the paper

- Part I: analysis of a choice problem of an efficient price-taking group.

  AGGREGATION OF PREFERENCES

- Part II: equilibrium state prices.

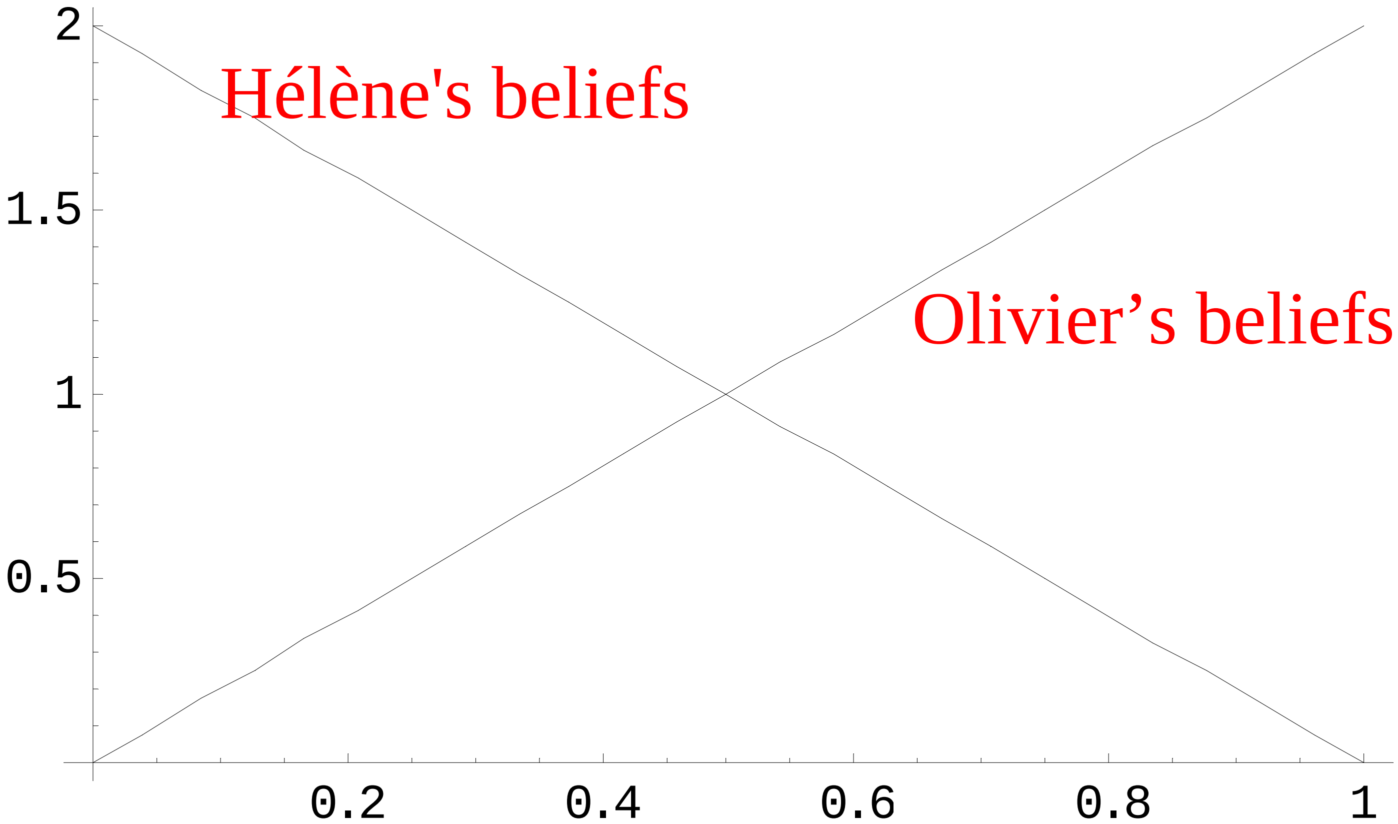
  THE EQUITY PREMIUM

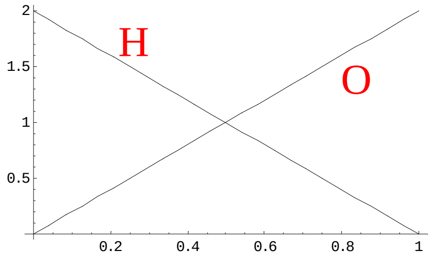
Part I
1. An illustration

A simple example

- The group has two equally-sized subgroups of agents, both with
  - the same constant relative risk aversion γ;
  - the same initial wealth.
- Some uncertainty on the state of nature.
- Disagreement on the density function.
- The group can purchase insurance, bet on specific states, purchase assets,...

Disagreement

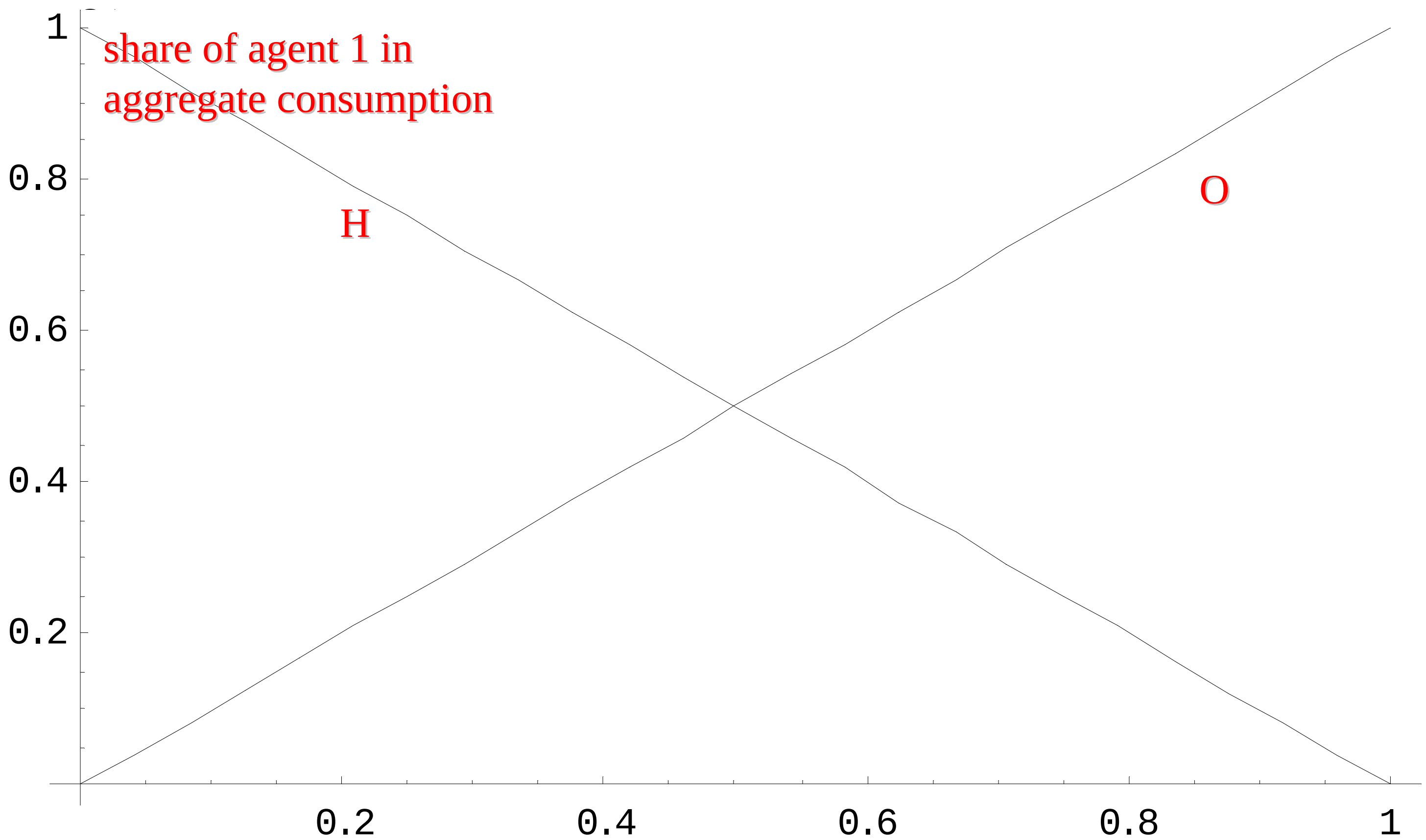




Aggregation of beliefs

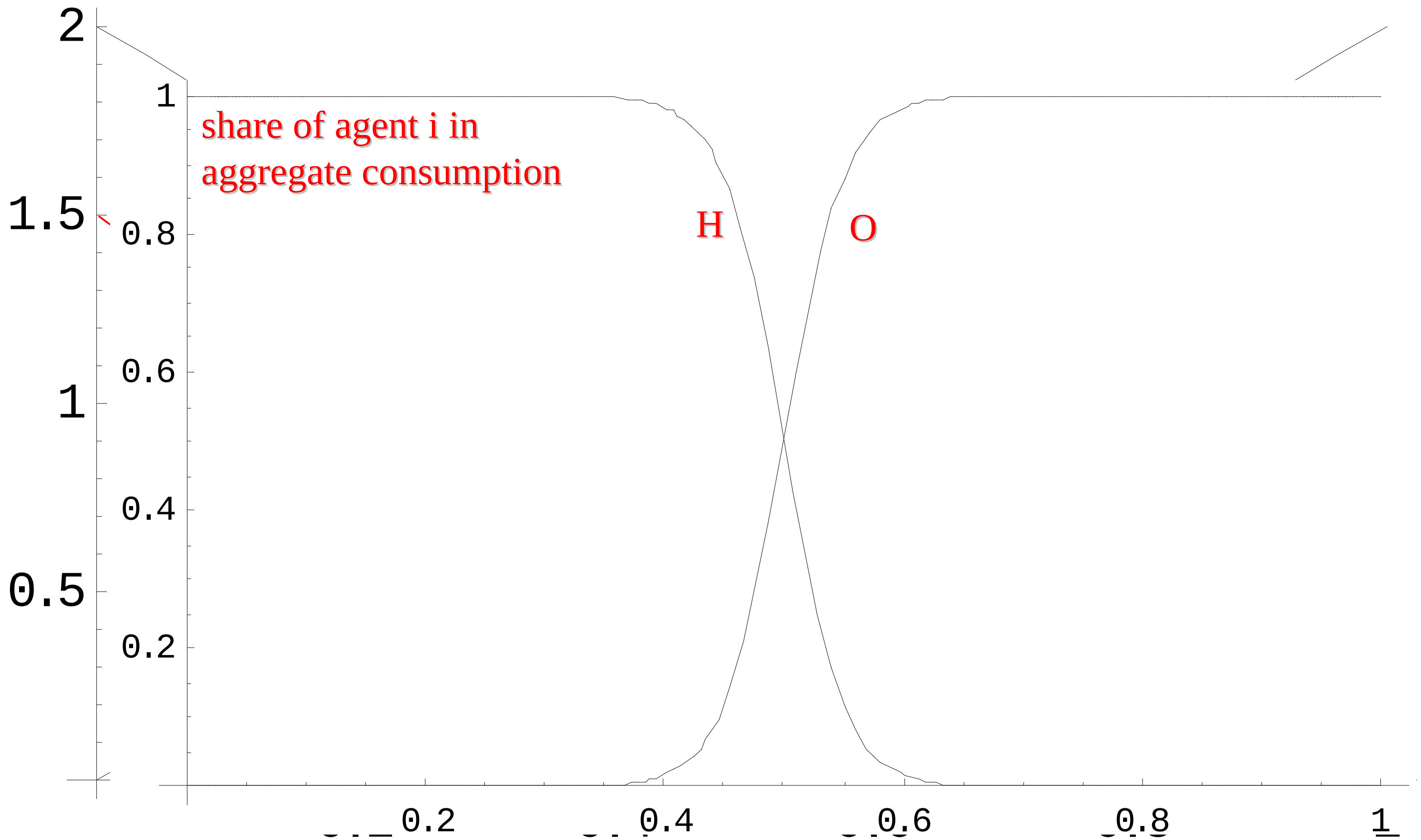
- I compute the competitive allocation of risk in the group:
  - Hélène will sell insurance for the high states to Olivier;
  - Olivier will sell insurance for the low states to Hélène.
- What is the attitude of the group towards marginal state-dependent transfers of wealth? I compute the preferences and beliefs of the representative agent.

Collective beliefs (γ=1)



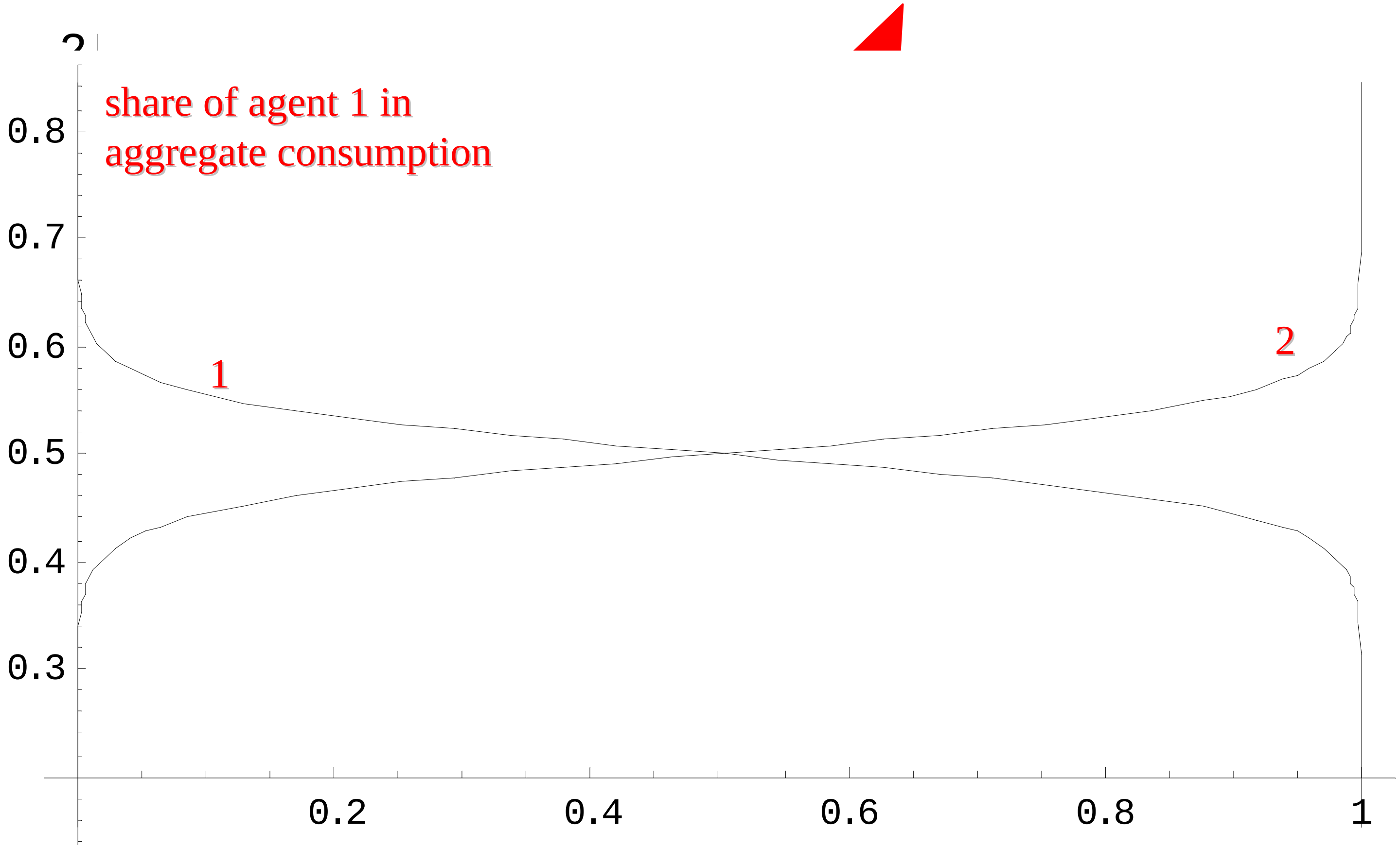
No effect of conflicts in beliefs on the collective beliefs

Collective beliefs (γ=0.1)



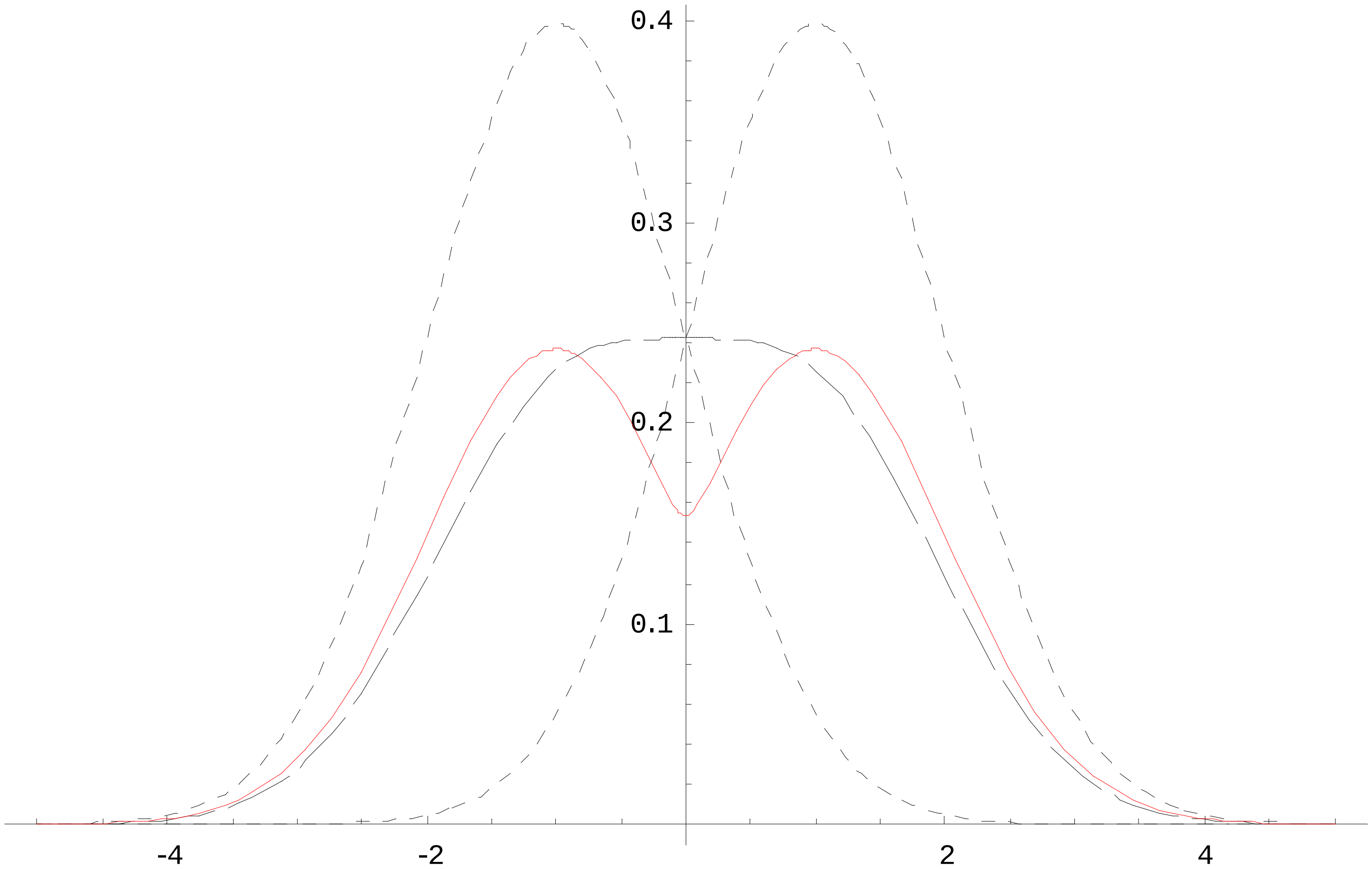
Disagreement increases the collective probability.

Collective beliefs (γ=10)

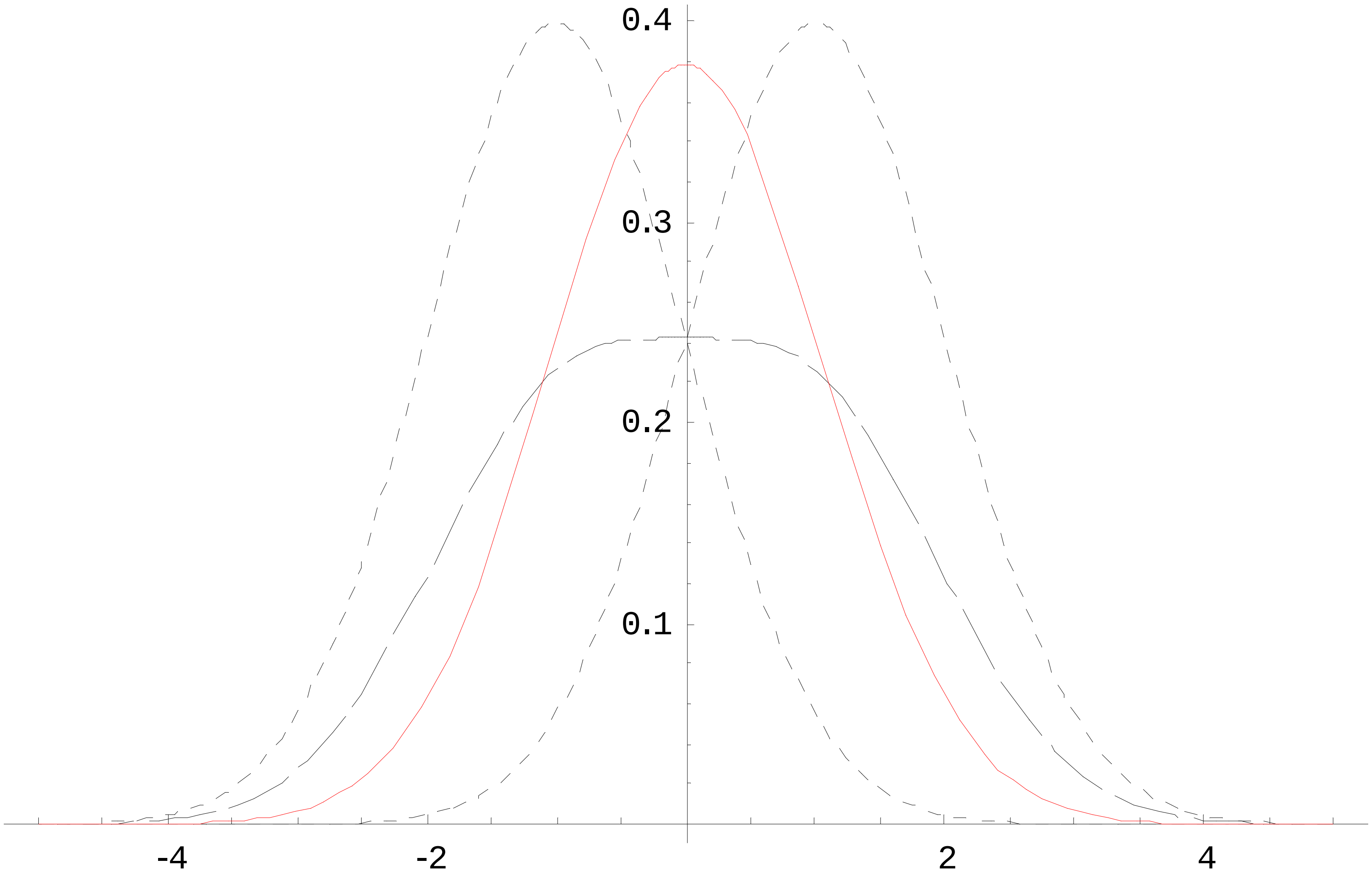


Disagreement reduces the collective probability.

Collective beliefs (γ=0.1)



Collective beliefs (γ=10)



2. The model

Collective decision problem

- Description of the environment:
  - S: set of states of nature.
- Characteristics of agent $\theta = 1,\ldots,N$:
  - A state-independent vNM utility: $u(c,\theta)$;
  - A probability/density function: $p(s,\theta)$;
  - A state-dependent endowment: $\omega(s,\theta)$.

Collective decision problem

- The group can transfer wealth across states:
  - Portfolio choice, insurance,...;
  - Prevention activities.
- $\pi(s)$: relative price of consumption in state s in S.
- $C(s,\theta)$= consumption of agent θ in state s.
- Budget constraint:

$$\int_S \pi(s)\left[\sum_{\theta=1}^{N} C(s,\theta) - \sum_{\theta=1}^{N} \omega(s,\theta)\right] ds = 0$$

The collective choice problem

$$\max_C \sum_{\theta=1}^{N} \lambda(\theta) \int_S p(s,\theta) u(C(s,\theta),\theta) ds$$

$$s.t. \quad \int_S \pi(s) \left(\sum_{\theta=1}^{N} \left[C(s,\theta) - \omega(s,\theta) \right] \right) ds = 0.$$

Additivity of (1) the SWF and (2) EU.

$z(s) = \Sigma C(s,\theta)/N$: mean consumption.

$P(s) = (p(s,1),..., p(s,N))$: vector of probabilities.

Remark

- We are going to derive properties of the collective beliefs that holds for all Pareto-efficient allocations within the group.
- In particular, these properties must hold for the competitive allocation.

3. The main results

The representative agent

$$\max_C \sum_{\theta=1}^{N} \lambda(\theta) \int_S p(s,\theta) u(C(s,\theta),\theta) ds$$

$$s.t. \quad \int_S \pi(s) \left(\sum_{\theta=1}^{N} [C(s,\theta) - \omega(s,\theta)] \right) ds = 0.$$

$$v(z,P) = \max_{c(z,P,.)} \sum_{\theta=1}^{N} \lambda(\theta) p(\theta) u(c(z,P,\theta),\theta)$$

$$s.t. \quad \frac{1}{N} \sum_{\theta=1}^{N} c(z,P,\theta) = z.$$

The cake sharing problem

$$\max_{z(.)} \int_S v(z(s),P(s)) ds$$

$$s.t. \quad \int_S \pi(s) [z(s) - \omega(s)] ds = 0.$$

The collective risk exposure problem

Result 1

$$\max_{z(.)} \int_S v(z(s), P(s))ds$$
$$s.t. \quad \int_S \pi(s)\left[z(s) - \omega(s)\right]ds = 0.$$

- *v(z(s),P(s))*: contribution of state *s* to ex-ante welfare.
- In general, $v(z, P) \neq p^v(P)h(z)$.
- Except in the ISHARA case!
- In the other cases, the utility function of the representative agent is state-dependent.
- The collective risk attitude is observationally equivalent to the collective attitude of a group with homogenous beliefs, but with a state-dependent utility function.

Result 2: The allocation of the aggregate risk

$$T(c,\theta) = -u'(c,\theta)/u''(c,\theta): \text{ absolute risk tolerance}$$

$$\frac{\partial c}{\partial z}(z,P,\theta) = \frac{T(c(z,P,\theta),\theta)}{N^{-1}\sum_{i=1}^{N} T(c(z,P,i),i)}$$

The share of the aggregate risk borne by an agent is proportional to his absolute risk tolerance.

Result 3: Aggregation of beliefs

$$R(z,P,\theta) = \frac{d \ln v_z(z,P)}{d \ln p(\theta)}$$

$R(z,P,\theta)$ = the percentage increase of the collective probability when the subjective probability of agent θ is increased by 1%.

The share of an agent's beliefs in the collective beliefs is proportional to his absolute risk tolerance.

Increasing disagreement

- Assumption: $u(c,\theta)=u(c)$.
- We are going to compare two states s and s'.

$$P(s) = (p(s,1),\ldots, p(s,N))$$
$$P(s') = (p(s',1),\ldots, p(s',N))$$

- There is more disagreement about s' than about s if the individual probabilities of state s' are "more dispersed" than for state s.

The geometric mean approach

$$p^v(P) = a\left[\prod_{\theta=1}^{N} p(\theta)\right]^{1/N}$$

$$\log p^v(P) = \log a + \frac{1}{N}\sum_{\theta=1} \log p(\theta)$$

Efficient aggregation rule under CARA

- Suppose that the log probabilities
  – have the same mean in s and s';
  – are more dispersed (MPS) in s' than in s.
- Absolute risk aversion: *-u''(c)/u'(c)*.

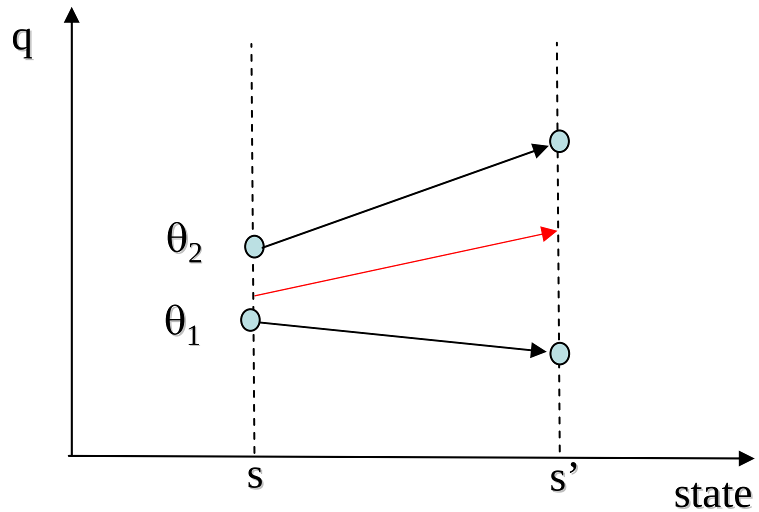
Result 5: error of the geometric mean approach

- Proposition: *The following two conditions are equivalent:*
  1. *Any mean-preserving spread in log individual probabilities raises the collective probability;*
  2. *Absolute risk aversion is decreasing.*

Intuition

- DARA: θ_2 is more risk-tolerant than θ_1.
- The collective beliefs is biased in his favor.

  $R(z,P(s),\theta_2)>1/2.$
- The collective probability goes upward.



$$\Delta \log v'(z,P) = R_1 \Delta \log p_1 + R_2 \Delta \log p_2$$

$$0.5 \Delta \log p_1 + 0.5 \Delta \log p_2 = 0$$

The arithmetic mean approach

$$p^v(P) = a\frac{1}{N}\sum_{\theta=1}^{N}\lambda(\theta)p(\theta)$$

Efficient aggregation rule under the log.

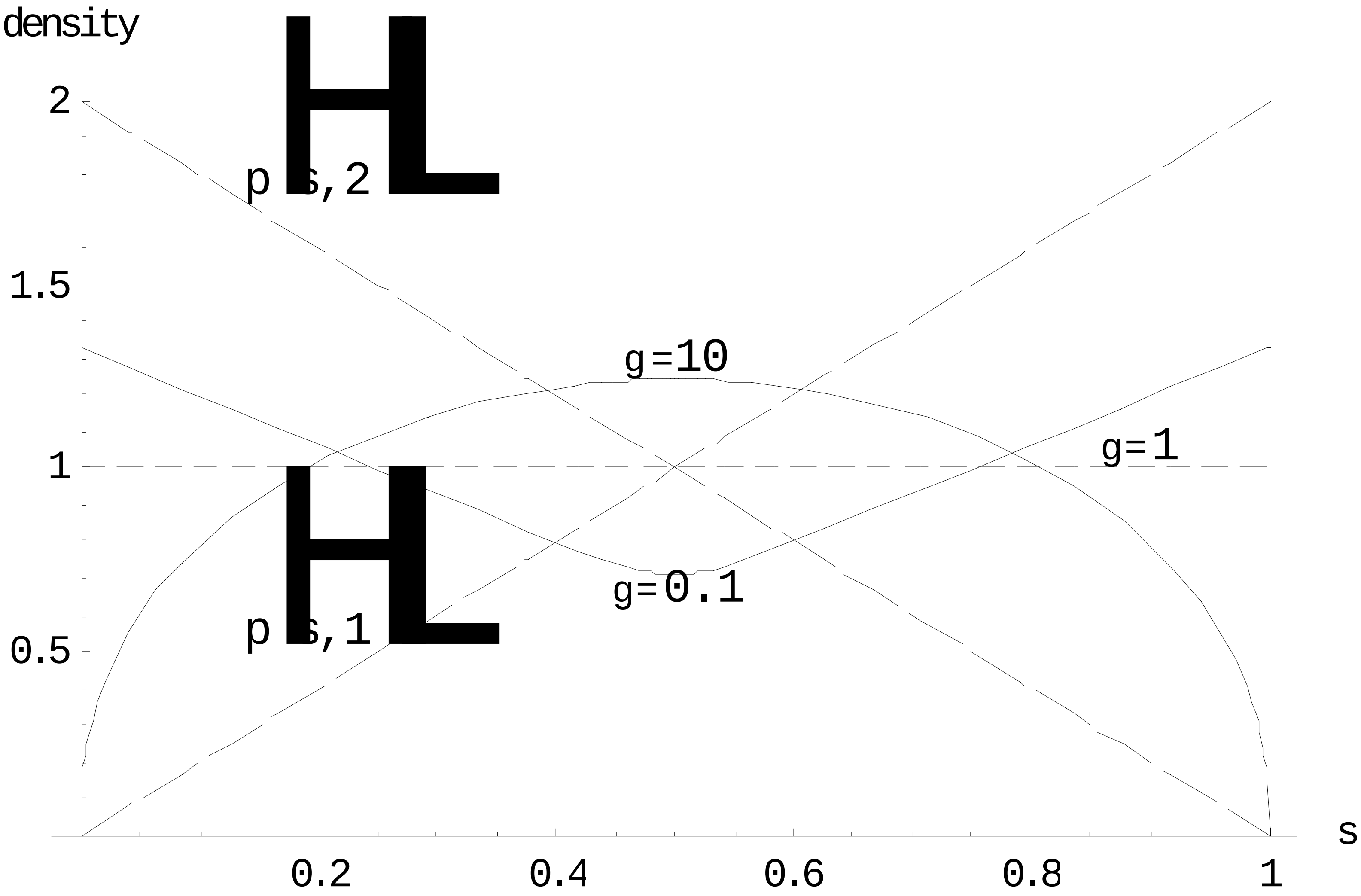
- Compare s' to s.
- Suppose that there is more disagreement in s' than in s.
- Suppose that the arithmetic means of individual probabilities are the same in s and s': MPS in probabilities.
- Absolute prudence: -*u'''(c)/u''(c)*.

Result 6: error of the arithmetic mean approach

- Proposition: *The following two conditions are equivalent:*
  1. *Any mean-preserving spread in individual probabilities reduces the collective probability;*
  2. *The absolute prudence is uniformly smaller than twice the absolute risk aversion.*

- *Special case CRRA: condition 2 is equivalent to relative risk aversion being larger than unity.*

Illustration with CRRA



Intuition

- $P<2A$.
- Under the veil of ignorance, more disagreement can be interpreted as more risk. How does it affect the marginal value of wealth?
- Precautionary effect: raises marginal value ÷ P.
- Rebalancing consumption towards less pessimistic agents. Similar to an increase in wealth: reduces marginal value ÷ A.

The two-state case

Remark

- In the two-state case, the degree of disagreement must be symmetric in the two states by construction.
- Previous results based on the notion of increasing disagreement are useless here.

A simple result

- Suppose that $P<2A$.
- Suppose that the distribution of individual probabilities is symmetric around its mean.
- Then, the collective probability of the less likely state is reduced.
- The efficient planner is an extremist!

Illustration

Two agents
p_1=10%
p_2=1%
CRRA