Whom should we believe? Collective risk-taking decisions with heterogeneous beliefs

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Abstract

Suppose that a group of agents having divergent expectations can share risks efficiently. We examine how this group should behave collectively to manage these risks. We show that the beliefs of the representative agent are in general wealth-dependent. We prove that the probability distribution used by the representative agent is biased in favor of the beliefs of the more risk tolerant agents in the group. From this central result, we show that increasing disagreement on the state probability raises the state probability of the representative agent. It implies that when most disagreements are concentrated in the tails of the distribution, the perceived collective risk is magnified. This can help to solve the equity premium puzzle.

Keywords: aggregation of beliefs, state-dependent utility, efficient risk sharing, disagreement, asset pricing, portfolio choices.

1 Introduction

People have divergent opinions on a wide range of subjects, from the growth rate of the economy next year, the profitability of a new technology to the risk of global warming. Suppose that this heterogeneity of beliefs does not come from asymmetric information but rather from intrinsic differences in how to view the world. People agree to disagree, which implies that prices and observed behaviors of other market participants do not generate any Bayesian updating of individual beliefs. We examine how the group as a whole will behave towards risk. Aggregating beliefs when agents differ on their expectations is useful to solve various economic questions, from asset pricing to cost-benefit analyses of collective risk prevention.

The attitude towards risk of a group of agents depends upon how risk is allocated in the group. For example, if an agent is fully insured by other agents, it is intuitive that this agent's beliefs should not affect the social welfare function. Only those who bear a share of the risk should see their expectations be taken into account on the collective risk decision. In this paper, we assume that risks can be allocated in a Pareto-efficient way in the group. In such a situation, the willingness to take risk is increasing in the Arrow-Pratt index of absolute risk tolerance. It implies that the beliefs of agents with a larger risk tolerance should have a larger impact on how individual expectations are aggregated. At the limit, those with a zero risk tolerance do not influence the group's expectations.

The properties of the socially efficient probability distribution are derived from the characteristics of the efficient allocation of risk in the group, such as the one derived from the competitive allocation with complete Arrow-Debreu markets. Borch (1960,1962), Wilson (1968) and Rubinstein (1974) were the first to characterize the properties of Pareto-efficient risk sharing. Wilson (1968), Lintner (1969), and more recently Calvet, Grandmont and Lemaire (2001), Jouini and Napp (2003) and Chapman and Polkovnichenko (2006), showed that the standard methodology of the representative agent can still be used when agents have heterogeneous beliefs. Leland (1980) examined the competitive equilibrium asset portfolios when agents have different priors on the distribution of state probabilities. Bossaerts, Ghirardato and Zame (2003) determine the equilibrium collective attitude towards risk when people have multiple priors and have different degrees of ambiguity aversion.

The cornerstone result of this paper is to compare two states of nature for which the distribution of individual probabilities are different. Consider for example a situation where all agents believe that state s_2 has the same probability of occurrence than another state s_1, except agent θ. Suppose that this agent has a subjective probability for s_2 that is 1 percent larger than for s_1. By how much should we increase the probability of state s_2 with respect to s_1 in the collective decision making? We show that the collective probability should be increased by x percents, where x is the percentage share of the aggregate risk that is borne by agent θ, or the agent θ's tolerance to risk expressed as a percentage share of the group's risk tolerance. More generally, the rate of change of the collective probability is a weighted mean of the rate of change of the individual probabilities. The weights are proportional to the individual risk tolerances. More risk tolerant agents see their beliefs better represented in the collective decision making under uncertainty. This intuitive result has several important consequences.

Observe first that, as initially observed by Hylland and Zeckhauser (1979),[1] the efficient aggregation of beliefs cannot be disentangled from the risk attitude of the group's members. Except in the case of constant absolute risk aversion, this individual risk attitudes depends upon the allocation of consumption in the group. It implies that the efficient collective probability distribution will be a function of the wealth per capita in the group. The representative agent has state-additive preferences as under the standard expected utility model, but the different terms of the sum cannot be written as a product of a probability that would depend only upon the state by a utility that would depend only upon consumption. Equivalently, this means that the representative agent has a state-dependent utility function, despite the fact that all members of the group have state-independent preferences.[2] Drèze (2001) and Drèze and Rustichini (2001) examine the effect of the state dependency of the utility function for risk management and risk transfers. Another way to interpret this result is that the collective probability distribution depends upon the aggregate wealth level. Wilson (1968) showed that this wealth effect vanishes only when agents have an absolute risk tolerance that is linear with the same slope. We reexamine this wealth effect when this

[1]See also Mongin (1995), Gilboa, Samet and Schmeidler (2001), and Gajdos, Tallon and Vergnaud (2005). Our work differs much from this branch of the literature by taking into account of risk-sharing opportunities within the group.

[2]Karni (1993) examines the problem of disentangling beliefs and tastes with state-dependent preferences.

condition is not fulfilled.

The efficient aggregation result for beliefs states that the rate of change of the collective probability between two states is a weighted mean of the rate of changes of individual probabilities. This result must be compared to the observation that the state probability used by the representative agent does not need to be in between the smallest and the largest state probabilities of the agents. Notice however that when deciding about transfers of wealth across states, what really matters are relative state prices per unit of probability. Thus the rate of change in the probabilities – or differences in log probabilities – is the relevant information for determining the collective risk exposure, and our aggregation formula provides exactly that information.

The main objective of the paper is to determine how the divergence of opinions about the true probability distribution of the states of nature affects the perception of risk by the representative agent, the optimal collective risk exposure, and the equilibrium asset prices. It is an immediate consequence of the independence axiom that the aggregation rule for beliefs is homogeneous of degree 1. When comparing two states of nature, if all individual probabilities for the second state are k percents larger than those of the first state, then the collective probability will also be k percents larger for the second state than for the first one. In other words, a uniform translation of individual log probabilities yields an equivalent translation in the collective probability. On this basis, we say that there is a relative increase in disagreement between two states if the distribution of individual *log* probabilities is more dispersed in one state than in the other. If this relative increase in disagreement preserves the mean log probabilities, we show that it raises the collective probability if and only if absolute risk aversion is decreasing (DARA). To illustrate, suppose that Mrs Jones has a larger subjective probability for a flood to occur this year than Mr Jones. Compared her own beliefs about floods, Mrs Jones has a subjective probability for the risk of an earthquake that is k percents larger, whereas Mr Jones has a subjective probability for an earthquake that is k percents smaller than his estimate of the probability of a flood. Thus, the mean log probabilities in the couple is the same for the two potential damages, but there is more disagreement about the likelihood of an earthquake than for a flood. Under DARA, it implies that, when Mr and Mrs Jones decide about their collective prevention efforts and insurance, they should use a larger probability of occurrence for an earthquake than for a flood.

Varian (1985) and Ingersoll (1987) answered to another related question: under which condition does a relative increase in disagreement that preserve the mean individual probability raises the collective probability? To illustrate, suppose again that Mr and Mrs Jones have a subjective flood probability of respectively p_{Mr} and $p_{Mrs} > p_{Mr}$. Suppose also that for earthquake, Mr Jones has a probability $p_{Mr} - k$, and Mrs Jones has a probability $p_{Mrs} + k$. Here, the arithmetic means of individual probabilities are the same for the two events but, as before, there is more relative disagreement for an earthquake than for a flood. Varian (1985) proved that, if relative risk aversion is larger than unity, the collective probability for an earthquake should be smaller than for a flood.

These results describe how the heterogeneity of beliefs affects the difference in collective probabilities for any pair of states. Going from this partial analysis to a more global one, it is necessary to describe the structure of disagreements across states. More precisely, it would be useful to determine whether conflicts of opinion in the population raise the risk perceived by the representative agent, in the sense of the first or second stochastic dominance order. If the answer to this question is positive, this could help to solve the equity premium puzzle, as explained by Cecchetti, Lam and Mark (2000) and Abel (2002). Contrary to us, they assume that all agents have the same beliefs that are biased with respect to what could be inferred from the existing data. In this paper, we endogenize the bias of the representative agent.

We use our main result to measure the impact of the heterogeneity of beliefs on the equity premium. We suppose that there are two groups of agents in the economy. Both groups believe that the growth rate of consumption per capita is lognormally distributed, but the optimistic group believe that the expected growth rate is larger than what is believed by the pessimistic group. We show that it implies that the relative degree of disagreement is increasing towards the extreme states. Because increasing relative disagreement raises the collective probability, we conclude that such a conflict of opinions makes the tails of the distribution heavier. Because the representative agent perceives a riskier macroeconomic risk, the equity premium is increased. In a plausible simulation, the conflict of opinions multiplies the equity premium by 4. Calvet, Grandmont and Lemaire (2001) also examine the effect of heterogeneous beliefs on the equity premium. They are able to sign this effect when the relative risk aversion of the representative agent is decreasing with average wealth.[3] Jouini and Napp (2003) examine a continuous-time model with constant relative risk aversion, allowing them to discuss the effect of the heterogeneity of beliefs on the risk-free rate.

The structure of the paper is as follows. Section 2 is devoted to the description of the aggregation problem when agents have heterogeneous preferences and beliefs. In section 3, we solve the risk-taking decision problem of the representative agent, assuming that collective preferences are known. We show how to aggregate individual risk tolerances and individual beliefs in this framework in section 4. In section 5, we show how the aggregate wealth in the group affects the aggregation rule of individual probabilities. In section 6, we describe the social efficient aggregation rule in the special case of exponential utility functions, and we apply it when individual beliefs are all normally distributed. In section 7, we define our concept of increasing disagreement, and we determine its effect on the perception of risk by the representative agent. Section 8 is devoted to the analysis of the effect of the heterogeneity of beliefs on the equity premium. We compare our results to Varian (1985)'s one in section 9. Finally, we present concluding remarks in section 10.

2 The aggregation problem

We consider an economy or a group of N heterogeneous agents indexed by $\theta = 1, ..., N$. Agents extract utility from consuming a single consumption good. The model is static with one decision date and one consumption date. At the decision date, there is some uncertainty about the state of nature s that will prevail at the consumption date. There are S possible states of nature, indexed by $s = 1, ..., S$. Agents are expected-utility maximizers with a state-independent utility function $u(., \theta) : R \to R$ where $u(c, \theta)$ is the utility of agent θ consuming c. We assume that $u_c = \partial u / \partial c$ is continuously differentiable and concave in c. As in Calvet, Grandmont and Lemaire (2001), we focus on interior solutions. To guarantee this, we assume that

[3] Hara and Kuzmics (2001) obtained independently results about how to aggregate risk aversion when beliefs are homogeneous.

$\lim_{c \to 0} \partial u / \partial c = +\infty$ and that $\lim_{c \to +\infty} \partial u / \partial c = 0$.

We also assume that each agent θ has beliefs that can be represented by a vector $(p(1, \theta), ..., p(S, \theta))$, where $p(s, \theta) > 0$ is the probability of state s assumed by agent θ, with $\sum_{s=1}^{S} p(s, \theta) = 1$. Agents differ not only on their utility and beliefs, but also on their state-dependent wealth: $\omega(s, \theta)$ denotes the wealth of agent θ in state s.

The group must take a decision towards a collective risk. This can be a portfolio choice, or a decision to invest in a prevention activity to reduce a global risk. In any case, the problem is to transfer wealth across states at an exogenously given exchange rate. The standard paradigm to analyze this problem is the Arrow-Debreu framework. We assume that there is a complete set of Arrow-Debreu securities in the economy. The equilibrium price of the Arrow-Debreu security associated to state s is denoted $\pi(s) > 0$. It means that agent θ must pay $\pi(s)$ ex-ante to receive one unit of the consumption good if and only if state s occurs. Because our model is static, we can normalize prices in such a way that $\sum_{s=1}^{S} \pi(s) = 1$.

A consumption plan is described by a function $C(., .)$ where $C(s, \theta)$ is the consumption of agent θ in state s. The consumption per capita in state s is denoted $z(s)$:

$$\frac{1}{N} \sum_{\theta=1}^{N} C(s, \theta) = z(s) \tag{1}$$

for all $s = 1, ..., S$. The mean initial endowment is denoted $\omega(s) = \sum_{\theta=1}^{N} \omega(s, \theta)/N$. When the group is active on contingent markets, z and ω need not be equal.

The crucial assumption of this paper is that the group can allocate risks efficiently among its members. An allocation C is Pareto-efficient if it is feasible and if there is no other feasible allocation that raises the expected utility of at least one member without reducing the expected utility of the others. This is a very general assumption. A special case is the competitive solution, which will be examined in section 9. In this paper as in Wilson (1968), we characterize the properties of *all* Pareto-efficient allocations. For a given vector of positive Pareto weights $(\lambda(1), ..., \lambda(N))$, normalized in such a way that $N^{-1} \sum_{\theta=1}^{N} \lambda(\theta) = 1$, the group would select the portfolio of Arrow-Debreu securities and the allocation of the risk within the group that maximize the weighted sum of the members' expected utility under the feasibility constraint:

$$\max_{C} \sum_{\theta=1}^{N} \lambda(\theta) \sum_{s=1}^{S} p(s,\theta) u(C(s,\theta),\theta) \tag{2}$$

$$s.t. \quad \sum_{s=1}^{S} \pi(s) \sum_{\theta=1}^{N} \frac{C(s,\theta)-\omega(s,\theta)}{N} = 0. \tag{3}$$

3 The cake-sharing problem

In this section, we decompose the decision problem presented in the previous section into two problems: a cake-sharing problem and a portfolio choice problem. We begin with the cake sharing problem. Consider a specific wealth per capita z and a vector $P = (p(1), ..., p(N))$ of N positive scalars. For this pair (z, P), define the following cake-sharing problem:

$$v(z,P) = \max_{x(.)} \sum_{\theta=1}^{N} \lambda(\theta) p(\theta) u(x(\theta),\theta) \quad s.t. \ \frac{1}{N} \sum_{\theta=1}^{N} x(\theta) = z, \tag{4}$$

The solution of this program is denoted $x^*(.) = c(z, P, .)$. The interpretation of this program is straightforward. A cake of size Nz must be shared among the N members of the group. The sharing rule is selected in order to maximize a weighted sum of the individual utility functions. In this well-behaved cake-sharing problem, z represents the consumption per capita, and $v(z, P)$ is the maximum sum of the members' utility weighted by the product of the Pareto weights $(\lambda(1), ..., \lambda(N))$ and the vector P. Our notation makes explicit that the efficient allocation c and the value function v depend upon the vector P. Notice that by construction, v is homogeneous of degree 1 with respect to P.

The other problem is a collective portfolio problem in which the group selects the state-dependent sizes of the cake that maximizes the sum of v across the states:

$$\max_{z(.)} \sum_{s=1}^{S} v(z(s), P(s)) \quad s.t. \ \sum_{s=1}^{S} \pi(s) \left(z(s) - \omega(s)\right) = 0, \tag{5}$$

where $P(s) = ((p(s,1),...,p(s,N))$ is the vector of subjective state probabilities across agents. The interpretation of this problem is also straightforward. An agent – hereafter referred to as the "representative agent" of the group – with initial endowment $(\omega(1),...,\omega(S))$ selects a portfolio of Arrow-Debreu securities $(z(1),...,z(S))$ that maximizes the ex-ante objective function $\sum_{s=1}^{S} v(z(s), P(s))$ subject to the standard budget constraint.

Obviously, combining these two-stage problems generates the solution to program (2), with $C(s,\theta) = c(z(s), P(s), \theta)$. Our solution strategy is thus simple. In section 4, we will show which characteristics of the v function are useful to characterize the efficient collective risk exposure that solves (5). In the remaining sections, we will link these properties of the v function to the primitive characteristics of individual preferences and beliefs. This will be done by focusing on the cake-sharing problem that defines the v function. Notice that this function describes the risk attitude and beliefs of the representative agent in the sense of Constantinides (1982). As we will see latter on, it is not true in general that the representative agent has preferences and beliefs that are multiplicatively separable as in the standard expected utility model. In other words, the representative agent has beliefs that are wealth-dependent, or equivalently, he has in general a state-dependent utility function. Following Wilson (1968), v will hereafter be referred to as the valuation function. We summarize the findings of this section in the following proposition.

Proposition 1 *Consider an economy characterized by price kernel $\pi(.)$, and a group of risk-averse expected-utility-maximizing agents $\theta = 1,...,N$ characterized by their concave utility function $u(.,\theta)$, beliefs $p(.,\theta)$ and endowment $\omega(.,\theta)$. Suppose that the group allocates risk in a Pareto-efficient way by using Pareto weights $\lambda(.)$. There exists a valuation function $v : R^+ \times [0,1]^N$ such that the aggregate optimal portfolio $z(.) = N^{-1} \sum_{\theta=1}^{N} C(.,\theta)$ solves program (5). This valuation function is defined by the cake-sharing program (4).*

In the following section, we exhibit the properties of the valuation function v that drive the optimal risk exposure.

4 Efficient collective risk exposure

In this section, we examine the determinants of the efficient collective risk exposure, assuming that the valuation function v is known. We postpone the aggregation problem to the next section. The optimal collective risk exposure solves problem (5). A state of nature is defined by the state price π and by the vector P of individual probabilities associated to that state. The optimal collective risk exposure can thus be characterized by a function $Z(\pi, P)$ that must satisfy the following first-order condition:[4]

$$v_z(Z(\pi, P), P) = \xi\pi, \tag{6}$$

for all (π, P), where ξ is the Lagrange multiplier associated to the budget constraint of program (5). The optimal solution to program (5) is then described as $z(s) = Z(\pi(s), P(s))$ for all $s \in S$. We now characterize the properties of Z. To do this, we first define function T^v as the absolute risk tolerance of the group, i.e.,

$$T^v(z, P) = -\frac{v_z(z, P)}{v_{zz}(z, P)}. \tag{7a}$$

Function $R(z, P, \theta)$ is the elasticity of the marginal valuation $v_z(z, P)$ to change in $p(\theta)$:

$$R(z, P, \theta) = \frac{\partial \ln v_z(z, P)}{\partial \ln p(\theta)} = \frac{p(\theta)v_{zp(\theta)}(z, P)}{v_z(z, P)}. \tag{8}$$

Because v_z is homogeneous of degree 1 in P by definition, it must be that

$$\sum_{\theta=1}^{N} R(z, P, \theta) = 1.$$

We consider the effect on Z of a marginal change of the state price π and of the vector P of individual probabilities. Fully differentiating the first-order condition (6) and eliminating the Lagrange multiplier yields the following result.

[4]The second-order condition is satisfied because v is concave in z.

Proposition 2 *Given the valuation function $v : R^+ \times [0,1]^N$ of the representative agent, the optimal aggregate portfolio $z(.)$ is characterized by a function $z(s) = Z(\pi(s), P(s))$. The optimal collective portfolio risk is measured by*

$$dZ = T^v(Z,P)\left[\sum_{\theta=1}^{N} R(z,P,\theta)\frac{dp(\theta)}{p(\theta)} - \frac{d\pi}{\pi}\right]. \qquad (9)$$

Equation (9) describes how the optimal consumption per capita varies across states as a function of the distribution of individual beliefs and of state prices. Seen from ex-ante, it describes the efficient risk exposure of the group. We see from (9) that the optimal collective risk exposure is proportional to the degree of absolute risk tolerance of the group which is measured by T^v.

When all agents have the same beliefs, equation (9) simplifies to

$$dZ = T^v(Z,p)\left[\frac{dp}{p} - \frac{d\pi}{\pi}\right]. \qquad (10)$$

When π is proportional to p, asset prices are actuarial, and full insurance is optimal (Z is constant). On the contrary, when $\pi(s')/p(s')$ is larger than $\pi(s)/p(s)$, state s' is more expensive than state s. It is then optimal to consume less in s' than in s. When agents have different beliefs, equation (9) tells us how to aggregate beliefs. Function R tells us how a change in the subjective probability of agent θ affects the demand for the corresponding state consumption. Namely, an increase of the subjective probability of agent θ by k percents has the same effect on the collective demand for consumption as a uniform increase of individual probabilities by Rk percents. *Thus, by definition, $R(z,P,\theta)$ measures the weight of agent θ's beliefs in the formation of aggregate beliefs.* By analogy to the homogeneous case (10), $\sum_\theta R d\ln p$ can be interpreted as a change in the collective probability derived from a change in the distribution of individual probabilities.

Keeping in mind equation (9), the remainder of the paper is devoted to this characterization of $T^v(z,P)$ and $R(z,P,\theta)$ of the collective valuation function v, which is defined by the cake-sharing program (4). Its first-order condition is written as

$$\lambda(\theta)p(\theta)u_c(c(z,P,\theta),\theta) = \psi(z,P) = v_z(z,P), \qquad (11)$$

for all (z,P), and for all $\theta = 1,...,N$, where ψ is the Lagrange multiplier associated to the feasibility constraint of program (4). The second equality comes from the envelop theorem.

5 The aggregation rules

In this section, we characterize the group's degree of tolerance to risk on the wealth per capita z and the group's beliefs as functions of the primitives of the model, i.e., the set of individual utility functions $u(.,\theta)$ and beliefs $p(.,\theta)$.

The collective attitude towards risk depends upon how this collective risk is allocated to the members' risk on consumption. This is characterized by $\partial c/\partial z$. Fully differentiating first-order condition (11) with respect to z and using the feasibility constraint $\sum_{\theta=1}^{N} c(z,P,\theta)/N = z$ yields the following well-known Wilson (1968)'s result:

$$\frac{\partial c}{\partial z} = \frac{T^u(c(z,P,\theta),\theta)}{N^{-1}\sum_{\theta'=1}^{N} T^u(c(z,P,\theta'),\theta')}, \tag{12}$$

where $T^u(c,\theta) = -u_c(c,\theta)/u_{cc}(c,\theta)$ is the absolute risk tolerance of agent θ. One can interpret this property of the efficient risk-sharing rule as follows: suppose that there are two states of nature that are perceived to be identical by all agents ($p(s,\theta) = p(s',\theta)$ for all θ), expect for the mean income z. Equation (12) shows how to allocate the collective wealth differential in the two states. Observe that the positiveness of the right-hand side of (12) means that individual consumption levels are all comonotone. More risk-tolerant agents should bear a larger fraction of the collective risk.

From this efficient collective risk-sharing rule, it is easy to derive the degree of risk tolerance of the group as a whole. Wilson (1968) obtains that

$$T^v(z,P) = \sum_{\theta'=1}^{N} T^u(c(z,P,\theta'),\theta'). \tag{13}$$

The group's absolute risk tolerance is the mean of its members' tolerance. There is no bias in the aggregation of individual risk tolerances. We conclude that this rule already valid in the simpler Wilson's model is robust to the introduction of heterogeneous expectations.

In the classical case with homogeneous beliefs, an important property of any Pareto-efficient allocation of risk is the so-called mutuality principle. It states that efficient individual consumption levels depend upon the state only through the wealth per capita z. Its economic interpretation is that all diversifiable risks are eliminated through sharing. In this classical case, the wealth level per capita z is a sufficient statistic for efficient individual consumption levels. The mutuality principle is obviously not robust to the introduction of heterogeneous beliefs because efficient allocation plans $c(z, P, \theta)$ depend also upon the distribution of individual subjective probabilities associated to the state. For example, agents will find mutually advantageous exchanges of zero-sum lotteries in order to gamble on states that they believe to be more likely than their counterpart. In the following, we examine the effect of a change in the distribution of individual probabilities P on the allocation of wealth and on the marginal valuation v_z.

The aggregation of beliefs cannot be disentangled from how the heterogeneity of beliefs affects the allocation of risk in the group. In the following proposition, we derive altogether the aggregation rule of beliefs and the allocation of diversifiable risks. The comparative exercise there and in the remainder of the paper consists in comparing two states of nature s and s' with $P(s') = P(s) + \Delta P$. It does *not* consist in increasing the subjective probability of state s by agent θ. We do not modify the structure of the beliefs in the economy.

Proposition 3 *The elasticity of the collective state probability to the subjective state probability of agent θ is proportional to agent's θ risk tolerance. More precisely, we have that*

$$R(z, P, \theta) = \frac{\partial \ln v_z(z, P)}{\partial \ln p(\theta)} = \frac{T^u(c(z, P, \theta), \theta)}{T^v(z, P)}, \tag{14}$$

where function T^v is defined in (13). The efficient allocation of consumption satisfies the following condition:

$$\frac{\partial c(z, P, \theta)}{\partial \ln p(\theta')} = \begin{cases} T^u(c(z, P, \theta), \theta)\left[1 - \dfrac{T^u(c(z, P, \theta), \theta)}{T^v(z, P)}\right] & if \quad \theta = \theta' \\ -\dfrac{T^u(c(z, P, \theta), \theta)T^u(c(z, P, \theta'), \theta')}{T^v(z, P)} & if \quad \theta \neq \theta'. \end{cases} \tag{15}$$

Proof: Fully differentiating the first-order condition (11) with respect to $p(\theta')$ and dividing both side of the equality by $\lambda p u_c = \psi$ yields

$$dc(z, P, \theta) = -T^u(c(z, P, \theta), \theta)\frac{d\psi}{\psi} \tag{16}$$

for all $\theta \neq \theta'$, and

$$dc(z,P,\theta') = T^u(c(z,P,\theta'),\theta')\left[\frac{dp(\theta')}{p(\theta')} - \frac{d\psi}{\psi}\right]. \tag{17}$$

By the feasibility constraint, it must be that $\sum_{\theta=1}^{N} dc(z,P,\theta) = 0$. Replacing $dc(z,P,\theta)$ by its expression given above allows us to rewrite this equality as

$$\frac{d\psi}{\psi} = \frac{T^u(c(z,P,\theta'),\theta')}{T^v(z,P)}\frac{dp(\theta')}{p(\theta')}. \tag{18}$$

Combining (16), (17) and (18) yields (15). By the envelop theorem, we also know that $v_z(z,P) = \psi(z,P)$. It implies that

$$d\ln v_z(z,P) = \frac{d\psi}{\psi}. \tag{19}$$

Combining equations (18) and (19) yields property (14).■

Let us first focus on property (15). Ceteris paribus, an increase in the state probability by agent θ increases his efficient consumption and it reduces the consumption by all other members of the group. Ex-ante, this means that the members take risk on their consumption even when there is no social risk, i.e., when z is state independent. Agents take a long position on states that they perceive to have a relatively larger probability of occurrence relative to the other members of the group. This illustrates the violation of the mutuality principle. Notice that the size of these side bets among the members of the group is proportional to the members' risk tolerance. At the limit, if an agent θ has a zero tolerance to risk, it is not efficient for him to gamble with others in spite of the divergence of opinions in the group.

Condition (14) provides a nice characterization of the aggregation of individual beliefs in groups that can share risk efficiently. The weight R of agent θ's beliefs in the formation of aggregate beliefs is proportional to that agent's degree of absolute risk tolerance. Thus, the aggregation of individual beliefs is biased in favor of those agents who are more risk tolerant. Combining properties (14) and (12), we obtain that

$$R(z,P,\theta) = \frac{1}{N}\frac{dc(z,P,\theta)}{dz}. \tag{20}$$

The collective probability distribution is biased towards those who actually bear the collective risk in the group.

In the remainder of the paper, we use the aggregation rule (14) to derive properties of the collective probability distribution.

6 Wealth effect on the aggregation of beliefs

The fact that all members of the group have a multiplicatively separable valuation function $p(s,\theta)u(c,\theta)$ does not imply that the valuation function of the representative agent inherits this property from them. In other words, it is not necessarily true that $v(z,P)=p^v(P)h(z)$, where $p^v(P)$ could be interpreted as the collective probability of a state whose distribution of subjective state probabilities across agents is $P=(p(1),...,p(N))$, and $h(z)$ would be the utility of mean wealth z. This non-separability implies that the collective tolerance to risk T^v is a function of P, and that the aggregation rule R for beliefs depends upon wealth z. The equivalence between these two non-separability properties of the valuation function is expressed by the following equality:

$$\frac{\partial(-1/T^v(z,P))}{\partial p(\theta)}=\frac{\partial R(z,P,\theta)}{\partial z}.$$

When the valuation function is not multiplicatively separable, one can say that the representative agent has a state-dependent utility function, or equivalently, that its subjective probability distribution is sensitive to changes in aggregate wealth.

We start with a rephrasing of another Wilson's result where such a wealth effect does not exist. It corresponds to situations where the derivative of individual risk tolerances $\partial T^u/\partial c$ are all identical and consumption independent. The corresponding set of utility functions is referred to as ISHARA. A utility function has an Harmonic Absolute Risk Aversion (HARA) if its absolute risk tolerance is linear in consumption: $\partial T^u/\partial c(c,\theta)=1/\gamma(\theta)$ for all c. A set of utility functions satisfies the Identically Sloped HARA (ISHARA) property if their absolute risk tolerances are linear in consumption with the same slope: $\gamma(\theta)=\gamma$ for all θ. The set of utility functions that satisfies these conditions must be parametrized as follows:

$$u(c,\theta)=\kappa\left(\frac{c-a(\theta)}{\gamma}\right)^{1-\gamma} \tag{21}$$

These utility functions are defined over the consumption domain such that $\gamma^{-1}(\ c-a(\theta))>0$. When $\gamma>0$, parameter $a(\theta)$ is often referred to as the minimum level of subsistence. This preference set includes preferences with heterogeneous exponential utility functions $u(c,t,\theta)=-\exp(-A(\theta)c)$ when γ tends to $+\infty$, and $a(\theta)/\gamma$ tends to $-1/A(\theta)$. Taking $a(\theta)=0$ for all θ, it also includes the set of power (and logarithmic) utility functions with the same relative risk aversion γ for all θ.

Proposition 4 *The aggregation rule R for beliefs is independent of the wealth per capita in the group if and only if the members of the group have ISHARA preferences (21):*

$$\frac{\partial R(z,P,\theta)}{\partial z}=0\quad\forall(z,P,\theta)\quad\Longleftrightarrow\quad\frac{\partial T^u(c,\theta)}{\partial c}\text{ is independent of } c \text{ and } \theta.$$

Proof: Fully differentiating equation (14) with respect to z and using property (12) yields that $\partial R/\partial z$ evaluated at (z,P) has the same sign that

$$\frac{\partial T^u}{\partial c}(c(z,P,\theta),\theta)-\sum_{\theta'=1}^{N}\frac{T^u(c(z,P,\theta'),\theta')}{T^v(z,P)}\frac{\partial T^u}{\partial c}(c(z,P,\theta'),\theta'), \tag{22}$$

For ISHARA preferences, $\partial T^u/\partial c$ is a constant, which implies that the above expression is uniformly equal to zero, implying that R is independent of the per capita wealth in the group. Reciprocally, R independent of z implies that

$$\frac{\partial T^u}{\partial c}(c(z,P,\theta),\theta)=\sum_{\theta'=1}^{N}\frac{T^u(c(z,P,\theta'),\theta')}{T^v(z,P)}\frac{\partial T^u}{\partial c}(c(z,P,\theta'),\theta')$$

for all θ and P. This can be possible only if $\partial T^u/\partial c$ is independent of c and θ, which means that the group has ISHARA preferences.■

The ISHARA condition guarantees that R remains constant when the wealth level changes in the group. This result is equivalent to the property that efficient sharing rules are linear in z in ISHARA groups with homogeneous beliefs. In Appendix A, we derive an analytical solution to the aggregation problem when the ISHARA condition is satisfied.

In Proposition 4, we assumed that the derivative of individual absolute risk tolerances be identical across agents. In the next Proposition, we show that agents with a large sensitivity of risk tolerance to changes in consumption have a share R in the aggregation of beliefs that is increasing with wealth.

Proposition 5 *To each vector* (z, P), *there exists a scalar* m *belonging to* $[\min_\theta \partial T^u(c(z, P, \theta), \theta)/\partial c, \max_\theta \partial T^u(c(z, P, \theta), \theta)/\partial c]$ *such that*

$$\frac{\partial R(z, P, \theta)}{\partial z} \geq 0 \text{ if and only if } \frac{\partial T^u(c(z, P, \theta), \theta)}{\partial c} \geq m(z, P).$$

Proof: This is a direct consequence of the fact that $\partial R/\partial z$ has the same sign than (22), taking

$$m(z, P) = \sum_{\theta'=1}^{N} \frac{T^u(c(z, P, \theta'), \theta')}{T^v(z, P)} \frac{\partial T^u}{\partial c}(c(z, P, \theta'), \theta').\blacksquare$$

Agents with a large sensitivity of risk tolerance to changes in consumption are those who increase their bearing of the collective risk when the group's wealth increase. The result follows from the fact that the share R of agent $\theta' s$ beliefs in the aggregation of beliefs is proportional agent θ's share in the group's risk.

In the special case of utility functions exhibiting constant relative risk aversion (CRRA), viz. $u(c, \theta) = c^{1-\gamma(\theta)}/(1 - \gamma(\theta))$, there is a negative relationship between relative risk aversion $\gamma(\theta)$ and $\partial T^u(c, \theta)/\partial c = 1/\gamma(\theta)$. Thus, the above Proposition applied in the case of CRRA utility functions means that *less risk-averse agents have a share R in the aggregation of beliefs that is increasing with wealth.* To illustrate, suppose that there are two agents in the group, respectively with constant relative risk aversion $\gamma(\theta_1) = 1$ and $\gamma(\theta_2) = 2$. In Figure 1, we have drawn the share R of agent θ_1's beliefs in the collective beliefs has a function of z, for P such that $\lambda(\theta_1)p(\theta_1) = \lambda(\theta_2)p(\theta_2)$. Because agent 1 is relatively less risk-averse than agent 2, this curve is upward sloping.

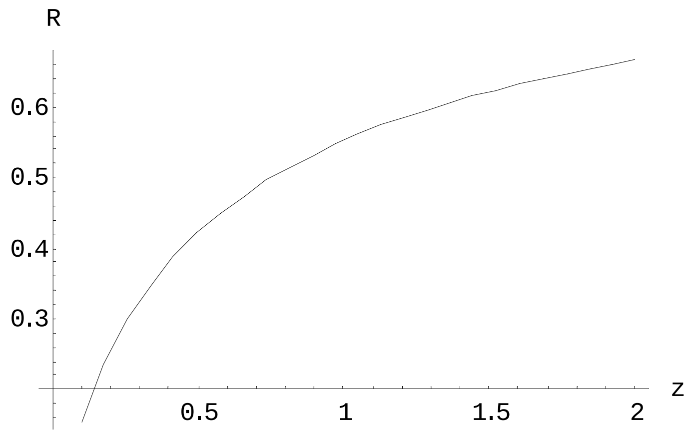

Figure 1: The share R of the beliefs of a low risk-averse agent in the collective beliefs as a function of the group's wealth per capita.

7 The socially efficient aggregation rule in the CARA case

As a benchmark, let us assume that absolute risk aversion is independent of consumption: $T^u(c,\theta) = a(\theta)$. As seen in the previous section, it implies that the valuation function $v(z, P)$ can be written as a product of a collective utility $h(z)$ by a collective probability $p^v(P)$. It implies also that equation (14) can be rewritten in this special case as

$$d \ln p^v(P) = \sum_{\theta=1}^{N} \frac{a(\theta)}{\sum_{\theta'=1}^{N} a(\theta')} d \ln p(\theta). \tag{23}$$

This yields

$$p^v(P) = K \prod_{\theta=1}^{N} p(\theta)^{\frac{a(\theta)}{\sum_{\theta'=1}^{N} a(\theta')}}, \tag{24}$$

where K is a normalizing constant that guarantees that $\sum_{s=1}^{S} p^v(P(s)) = 1$. In the CARA case, the efficient collective state probabilities are proportional to the geometric mean of the individual state probabilities. Aggregation rule (24) in the CARA case is due to Rubinstein (1974).

This aggregation rule is particularly easy to use when all individual beliefs are normally distributed. Suppose that agent θ, $\theta = 1, ..., N$, believes that states are normally distributed with mean $\mu(\theta)$ and variance $\sigma^2(\theta)$. A direct consequence of (24) is that the collective beliefs p^v are also normally distributed with mean

$$\mu^v = \frac{\sum_{\theta=1}^{N} \frac{a(\theta)\mu(\theta)}{\sigma^2(\theta)}}{\sum_{\theta=1}^{N} \frac{a(\theta)}{\sigma^2(\theta)}}, \tag{25}$$

and variance

$$\sigma^{v^2} = \left[\frac{\sum_{\theta=1}^{N} \frac{a(\theta)}{\sigma^2(\theta)}}{\sum_{\theta=1}^{N} a(\theta)} \right]^{-1}. \tag{26}$$

The collective mean is an average of the individual means weighted by the ratios $a/\sigma^2$of the individual absolute risk tolerance by the individual variance. The collective variance is the harmonic mean of the individual variances weighted by the corresponding individual risk tolerance. This very simple result due to Lintner (1969) can serve only as a benchmark, because there is a clear consensus in our profession that constant absolute risk aversion is not a realistic assumption.

8 The effect of increasing disagreement on the demand for Arrow-Debreu securities

In this section, we want to determine the effect of the divergence of opinions on the collective attitude towards risk. The efficient collective risk exposure is governed by how v_z fluctuates with the distribution of individual probabilities P. We want to determine the sign of $v_{zp(\theta)}(z, P)$, or to compare $v_z(z, P')$ to $v_z(z, P)$ in the large. If $v_z(z, P')$ is larger than $v_z(z, P)$, the demand for the contingent claim is increased in the state whose individual state probabilities is P' than in the state whose individual probabilities is P. Because the wealth per capita is constant in this comparative statics exercise, the increase in the marginal valuation v_z can be interpreted as an increase in the collective probability associated to the state with P'.

The effect of a shift in distribution on the collective probability depends upon the dispersion of individual beliefs in a complex way. As a benchmark, consider the proportional shift in distribution with $P' = (1 + k)P$. Each member of the group believes that state s' is k time more likely than state s. The decision problem (4) is unchanged by this multiplicative change in the parameters of the problem. The efficient allocation of z will be the same in the two states, and $v_z(z,(1+k)P) = (1+k)v_z(z,P)$, for all z and P. Thus, as stated before, v and its partial derivatives with respect to z are homogeneous of degree 1 in the vector of individual probabilities P. In the following, we define a family of shifts in P that are not proportional.

8.1 Relative increase in disagreement

In this subsection, we want to define a notion of increasing disagreement, keeping in mind that a purely multiplicative spread of individual probabilities has no effect on the collective probability. We hereafter define a concept of increasing disagreement that is based on the Monotone Likelihood Ratio (MLR) order. However, the main ingredient in this section is not the individual subjective probabilities $p(\theta)$, but rather the Pareto-weighted ones $q(\theta) = \lambda(\theta)p(\theta)$. We say that a marginal shift dP from an initial vector of individual probabilities P yields increasing disagreement if those agents with a larger initial $q(\theta)$ also have a larger *rate* of increase $d\ln q(\theta) = d\ln p(\theta)$. Compared to a proportional increase, the distribution of individual probabilities becomes more dispersed. Thus, there is an increase in disagreement relative to a proportional shift in individual probabilities.

Definition 1 *Consider a specific distribution of individual probabilities* $P = (p(1), ..., p(N))$ *and a specific Pareto-weight vector* $(\lambda(1), ..., \lambda(N))$. *We say that a marginal shift* dP *yields a relative increase in disagreement if* $q(\theta) = \lambda(\theta)p(\theta)$ *and* $d\ln q(\theta)$ *are comonotone: for all* (θ, θ') :

$$[q(\theta') - q(\theta)]\,[d\ln q(\theta') - d\ln q(\theta)] \geq 0. \tag{27}$$

In Appendix B, we show how to generalize our definition of increasing disagreement "in the small" to non-marginal changes of individual probabilities. The definition states that those with a larger subjective probability also have a larger rate of increase of their probability. If we assume without loss of generality that q is increasing in θ, this is equivalent to require that $p(\theta')/p(\theta)$ be increased by the shift whenever $\theta' > \theta$. This is a MLR property. Notice that our definition of increasing disagreement does not constrain in any way how the mean (log) (Pareto-weighted) probability is affected by the shift in distribution. In the following Proposition, we show that increasing disagreement generates a Rothschild-Stiglitz (1970) spread in the distribution of Pareto-weighted individual log probabilities.

Proposition 6 *Any marginal shift dP that preserves the mean $N^{-1}\sum_{\theta=1}^{N}\ln q(\theta)$ is a relative increase in disagreement if and only if it generates a Rothschild-Stiglitz spread of* $(\ln q(1), ..., \ln q(N))$.

Proof: See Appendix C.

To illustrate, let us consider two examples. There are two agents, $\theta = 1$ and $\theta = 2$. In the first example, there is a continuum of states of nature $s \in S = [0, 1]$. The beliefs of agent θ is represented by an exponential density function $p(s, \theta) = \delta_\theta \exp[\delta_\theta s]/(\exp[\delta_\theta] - 1)$. This means that agent θ has a constant rate of increase $\delta_\theta = d\ln p(s, \theta)/ds$ of his state probabilities . In Figure 2, we draw the two dashed curves $q(., 1)$ and $q(., 2)$ by assuming that $\delta_1 = -\delta_2 = 5$ and $\lambda(1) = \lambda(2)$. In this environment, the mean log individual probabilities is constant through the states of nature. Moreover, because

$$d\ln q(s,2) - d\ln q(s,1) = -2\delta_1 ds < 0 \quad \text{and} \quad q(s,2)-q(s,1)\begin{cases} > 0 & if \quad s < 0.5 \\ < 0 & if \quad s > 0.5, \end{cases}$$

increasing s at the margin everywhere between 0 and 0.5 decreases disagreement in the group, whereas marginally increasing s everywhere between 0.5 and 1 increases disagreement. Because the mean log probabilities is constant across states, this is also an example of Rothschild-Stiglitz increases/decreases in risk of $\ln q$.

In our second example described in Figure 3, the two agents believe that states are normally distributed with variance σ^2, but their beliefs differ on the mean $\mu(\theta)$, with $\mu(2) > \mu(1)$. Considering for simplicity the Pareto-allocation with $\lambda(1) = \lambda(2)$, we have that

$$q(s,2)-q(s,1) = \frac{1}{\sqrt{2\pi}\sigma}\left[e^{-\frac{(s-\mu(2))^2}{2\sigma^2}} - e^{-\frac{(s-\mu(1))^2}{2\sigma^2}}\right]\begin{cases} < 0 & if \quad s < 0.5(\mu(1)+\mu(2)) \\ > 0 & if \quad s > 0.5(\mu(1)+\mu(2)) \end{cases}$$

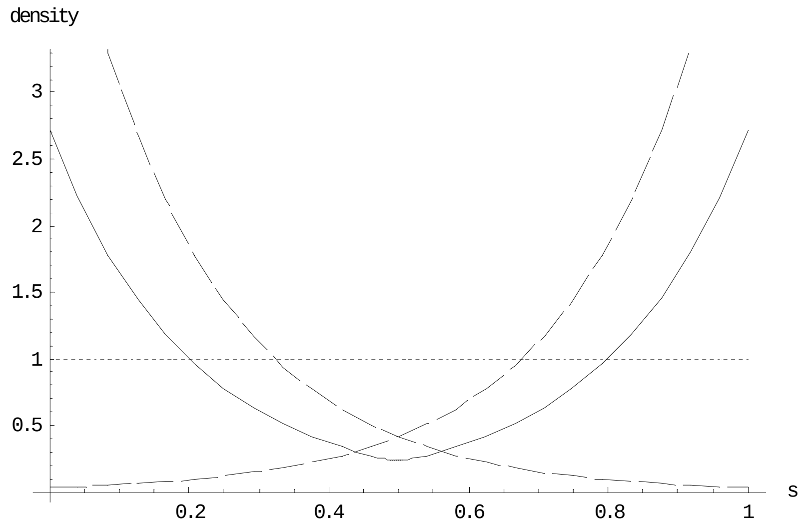

Figure 2: Heterogenous density functions with $p(s,1) = 5\exp[5s]/(\exp[5]-1)$ and $p(s,2) = -5\exp[-5s]/(\exp[-5]-1)$.

and

$$d\ln q(s,2) - d\ln q(s,1) = \frac{\mu(2)-\mu(1)}{\sigma^2}ds > 0.$$

It implies that a marginal increase in s yields a *relative* decrease in disagreement whenever $s < 0.5(\mu(1)+\mu(2))$, otherwise it yields a *relative* increase in disagreement. In these two examples, relative disagreement is the largest in the extreme states.

8.2 Our main results

In order to isolate the effect of heterogeneous beliefs, we hereafter assume that preferences are homogeneous in the population in the sense that all agents have the same utility function. Consider an initial distribution $P = (p(1),...,p(N))$ of individual probabilities, and a shift $dP = (dp(1),...,dp(N))$ in this distribution. Using Proposition 3 together with the assumption that

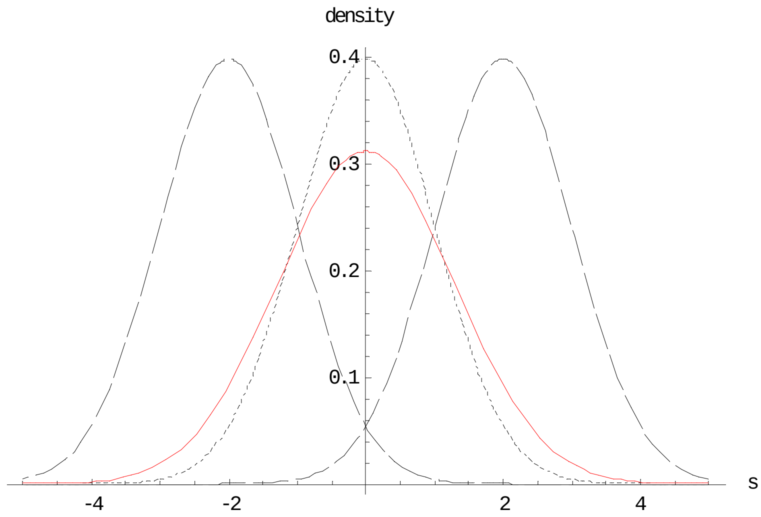

Figure 3: Normal heterogenous beliefs with $\sigma^2(1) = \sigma^2(2) = 1$, $\mu(1) = -2$ and $\mu(2) = 2$.

all agents have the same utility function, this can be rewritten as

$$d \ln v_z(z, P) = \sum_{\theta=1}^{N} \frac{T^u(c(z, P, \theta))}{T^v(z, P)} d \ln p(\theta), \tag{28}$$

assuming $dz = 0$. The left-hand side of this equality can be interpreted as the rate of increase in the collective probability. Equation (28) states that it is a weighted mean of the rate of increase in the individual probabilities. The weights are proportional to the individual absolute risk tolerance. As seen before, in the special case of constant and identical absolute risk aversion (ICARA), equation (28) can be rewritten as

$$d \ln v_z(z, P) = N^{-1} \sum_{\theta=1}^{N} d \ln p(\theta). \tag{29}$$

This means that relative increases in disagreement have no effect on the collective probability in the ICARA case. It implies for example that when the agents have exponential densities as in Figure 2, the (dotted) collective density function is uniform. When they have normal beliefs with the same variance as in Figure 3, the (dotted) collective beliefs is normal with a mean equaling the average individual means. In this section, we suppose that the social planner takes this geometric aggregation rule of individual beliefs into consideration, and we determine the error that it generates when the ICARA condition on individual preferences is not satisfied.

Suppose that, compared to a reference state, we examine an alternative state where the probability of Mr Jones is increased by $k+\varepsilon\%$ and the probability of Mrs Jones is reduced by $k-\varepsilon\%$. The couple using the geometric mean approach would increase the probability of this alternative state by $k\%$ without taking into account of their increased divergence of opinions. However, the socially efficient rule would increase the collective probability by ηk, where η is defined by

$$\eta(z,P,dP)=\frac{d\ln v_z(z,P)}{\frac{1}{N}\sum_{\theta=1}^{N}d\ln p(\theta)}.$$

When η is larger (smaller) than unity, using the geometric aggregation rule would underestimate (overestimate) the rate of increase of the collective probability. Thus $\eta-1$ measures the error of using the geometric rule.

We see that $\eta(z,P,dP)$ is larger than unity if and only if

$$N^{-1}\sum_{\theta=1}^{N}T^u(c(z,P,\theta))d\ln p(\theta)\geq\left[N^{-1}\sum_{\theta=1}^{N}T^u(c(z,P,\theta))\right]\left[N^{-1}\sum_{\theta=1}^{N}d\ln p(\theta)\right]. \quad (30)$$

When it is T^u is not constant, the allocation of consumption in the group will affect the weights in the aggregation formula (28). Suppose without loss of generality that $q(\theta)=\lambda(\theta)p(\theta)$ is increasing in θ. Combining the first-order condition (11) with risk aversion implies that $c(z,P,\theta)$ is increasing in θ. Under decreasing absolute risk aversion (DARA), it implies in turn that $T^u(c(z,P,\theta))$ is also increasing in θ. In short, the definition of increasing disagreement just guarantees that T^u and $d\ln p$ be comonotone under DARA. Applying the covariance rule to $E\left[T^u d\ln p\right]$ directly implies (30), or $\eta\geq 1$. Of course, switching to either increasing absolute risk aversion or decreasing disagreement would yield $\eta\leq 1$.

Proposition 7 *Suppose that the individual utility functions are identical. The following two conditions are equivalent:*

1. *For any wealth z, any initial distribution of individual probabilities P and any shift dP yielding a relative increase in disagreement, the rate of increase of the collective probability is larger than the mean rate of increase of individual probabilities: $\eta(z,P,dP)\geq 1$;*

2. *Absolute risk aversion is decreasing: $\partial T^u/\partial c\geq 0$.*

Proof: The sufficiency of DARA has been proved above. Suppose now by contradiction that T^u is locally decreasing in the neighborhood B of c_0. Then, take $z=c_0$ and an initial distribution $P(\varepsilon)$ such that $\lambda(\theta)p(\theta)=k+\varepsilon\theta$ for all θ. When $\varepsilon=0$, $c(z,P(0),\theta)=c_0$ for all θ. Take a small ε such that $c(z,P(\varepsilon),\theta)$ remains in B for all θ. By assumption, the shift dP exhibits increasing disagreement, which means that $c(z,P(\varepsilon),\theta)$ and $d\ln p(\theta)$ are comonotone. This implies that $T^u(c(z,P(\varepsilon),\theta))$ and $d\ln p(\theta)$ are anti-comonotone, thereby reversing the inequality in (30). This implies that DARA is necessary for property 1.■

Under DARA, a mean-preserving spread in log probabilities always raises the collective probability. The intuition of this result is easy to derive from the central property (14) of the aggregation of heterogeneous beliefs. Under DARA, this property states that those who consume more see their beliefs better represented in the aggregation. But by definition of an increase in disagreement, those who consume more are also those who have a larger rate of increase in their subjective probability. We conclude that, because of the bias in favor of those who consume more, an increase in disagreement raises the collective probability even when the mean rate of increase in individual probabilities is zero.

Proposition 7 provides a local result. We can also use it to obtain a result "in the large" about how heterogeneous beliefs transform the riskiness of the collective distribution. Suppose that, as in the two examples illustrated in Figures 2 and 3, the relative degree of disagreement be decreasing for small s, and increasing for larger ones. Under DARA, this implies that the slope of the efficient collective density is smaller than the slope of the collective density obtained by the geometric rule for small s, and that it is larger than it for larger s. In other words, the tails of the efficient collective density are heavier than the collective density obtained by the geometric rule.

Corollary 1 *Suppose that absolute risk aversion is decreasing. Suppose also that $S\subset R$ and that there exists $\widehat{s}$ such that an increase in s yields a relative* *decrease (resp. increase) in disagreement when s is smaller (resp. larger) than $\widehat{s}$. It implies that the efficient collective beliefs are a mean-preserving spread of the collective beliefs based on the geometric rule.*

When absolute risk aversion is increasing, or when relative disagreement is first increasing and then decreasing, the opposite result holds.

In order to illustrate Proposition 7 and the above corollary, let us reconsider the case represented in Figure 2 with two agents having symmetric exponential densities. Remember that the mean rate of increase in probabilities is uniformly zero in this example. Suppose that relative risk aversion is a constant $\gamma = 0.1$.[5] The U-curve in Figure 2 describes the collective density function in this case. Because when $s > 0.5$ a marginal increase in s yields a relative increase in disagreement, DARA implies that the collective probability is increasing in s in this region. The slope of the collective density function is very similar to the slope of the density function of agent $\theta = 1$. This is because most of the aggregate wealth is consumed by that agent in these states, as seen in Figure 4. This implies that the social planner who considers transferring wealth across states in this region will mostly be concerned by the beliefs of that agent. In region $s < 0.5$ on the contrary, the collective probability is decreasing in s because a larger s yields less relative disagreement. Most of the aggregate wealth is consumed by agent $\theta = 2$ in these states, which implies that the social planner who consider transferring wealth across these states will use beliefs whose sensitivity to changes in s is close to the one of the subjective density function of that agent. Comparing this solution to the CARA case where disagreements have no effect on the collective beliefs that would be uniform in that case, the collective density is heavier in the tails. In terms of collective risk management, a social planner using the geometric rule would not purchase enough insurance for the states with the largest relative degree of disagreement, i.e., extreme states in this example.

Let us now turn to the more interesting case illustrated by Figure 3, where the two types of agents have beliefs that are normally distributed with the same variance $\sigma^2 = 1$, $\mu(1) = -2$ and $\mu(2) = 2$. We know that the geometric aggregation rule – which is efficient in the CARA case – yield a (dotted)

[5]This unrealistically low level of relative risk aversion is selected for a pedagogical purpose. It yields frenzy side bets between the pessimistic and optimistic agents, as seen in Figure 4.

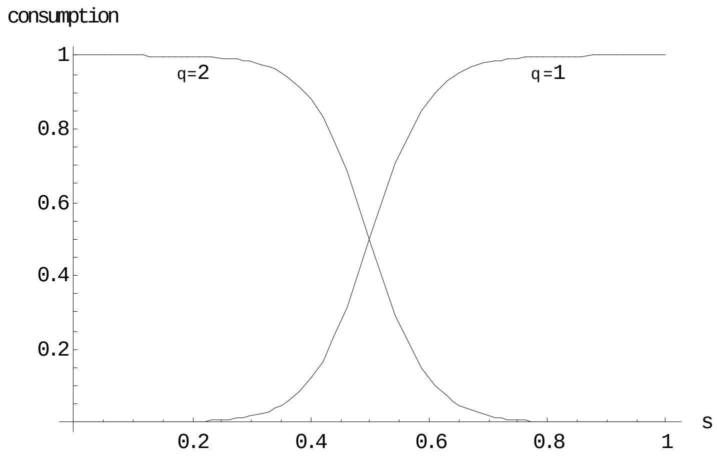

Figure 4: Optimal allocation of risk when $c(1, P(s), \theta)$, when $\lambda(1) = \lambda(2)$, $p(s, 1) = 5\exp[5s]/(\exp[5] - 1)$, $p(s, 2) = -5\exp[-5s]/(\exp[-5] - 1)$, and relative risk aversion equals $\gamma = 0.1$.

collective belief that is normal with a zero mean. Suppose alternatively that relative risk aversion is constant and is equal to $\gamma = 10$, yielding DARA. The plain bell curve in this figure describes the efficient collective density function in this case. We know that the relative degree of disagreement is decreasing for negative states, and that it is increasing for positive states. Proposition 7 implies that the slope of the collective density will be smaller than the slope of the "geometric" (dotted) density for negative states, and it will be larger than it for positive states. As in the previous example, because relative disagreement is larger in the tails, DARA implies that the collective density is heavier in the tails than under the geometric aggregation rule.

In Figure 5, we show how the efficient collective density is related to the degree of relative risk aversion γ. When relative risk aversion tends to infinity, absolute risk aversion becomes constant, and the collective beliefs become normal with a zero mean. On the contrary, when relative risk aversion is reduced, more and more side bets among optimistic and pessimistic agents become efficient. For very low γ, agent 1 consumes most of the cake in negative states, and agent 2 does the same for positive states. For the reasons explained above, it implies that the collective density will be almost

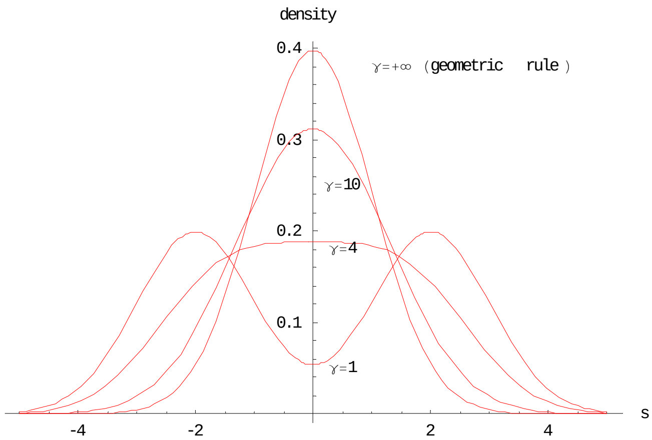

Figure 5: Efficient collective density functions when relative risk aversion γ is constant, and when both agents have normal beliefs with the same variance $\sigma^2 = 1$, assuming $\mu(1) = -2$, $\mu(2) = 2$.

proportional to $p(s,1)$ for negative states, and to $p(s,2)$ for positive states (see equation (40) in Appendix A). This is why we obtain collective density functions with two modes and heavy tails when γ is small.

These examples also illustrate another important feature of the aggregation of beliefs. Contrary to the intuition, the collective probability of any state s needs not to belong to the interval bounded by $\min_{\theta\in\Theta} p(s,\theta)$ and $\max_{\theta\in\Theta} p(s,\theta)$. This is in sharp contrast with the rate of increase in the collective probability, which is a weighted mean of the rate of increase in the individual probabilities, as stated by equation (28).

9 Asset pricing with heterogeneous beliefs

What are the implications of these results on the forward price of equity and the equity premium?We answer this question by reinterpreting the "group" that we considered earlier in this paper as the set of all consumers in an exchange economy à la Lucas (1978). This implies that the equilibrium condition is $\omega(s) = z(s)$ for all s, where $\omega(s)$ and $z(s)$ are respectively the mean endowment and the consumption per capita in state s. The first-order condition of program (5) can be rewritten as an equilibrium condition as follows:

$$v_z(\omega(s), P(s)) = \pi(s),$$

where $\pi(s)$ is the price of the Arrow-Debreu security associated to state s. We normalized the Lagrange multiplier to unity. Proposition 7 directly implies that the price of this asset is increasing in the relative disagreement of individual probabilities associated to the corresponding state. If there are two states with the same average log probability, the Arrow-Debreu security associated to the state with the larger degree of relative disagreement has a larger equilibrium price.

The price of equity equals

$$P^e = \frac{\sum_{s=1}^{S} \omega(s)\pi(s)}{\sum_{s=1}^{S} \pi(s)} = \frac{\sum_{s=1}^{S} \omega(s) v_z(\omega(s), P(s))}{\sum_{s=1}^{S} v_z(\omega(s), P(s))}.$$

Suppose that the ISHARA condition holds, which implies that $v_z(z, P) = p^v(P)h'(z)$. The representative agent then perceives an equity premium equaling

$$\phi = -1 + \frac{\left[\sum_{s=1}^{S} p^v(P(s))\omega(s)\right] \quad \left[\sum_{s=1}^{S} p^v(P(s))h'(\omega(s))\right]}{\sum_{s=1}^{S} p^v(P(s))\omega(s)h'(\omega(s))}.$$

We are interested in determining the effect of the heterogeneity of beliefs on this price of macroeconomic risk. Corollary 1 shows that when relative disagreements are concentrated in the tails, the representative agent perceives a distribution of states that is more dispersed than the distribution generated by using the intuitive geometric aggregation rule. In Abel (2002)'s terminology, the representative agent has some doubts about the distribution of equity returns. If ω is monotone in s, this increased dispersion in the state space S corresponds to an increased perceived dispersion in consumption, i.e., to an increased perceived macroeconomic risk. Because of risk aversion, this should induce a reduction in the demand for equity. Eventually, at equilibrium, this should yield a reduction in the price of equity, and to an increase in the equity premium ϕ.

The idea that a mean-preserving spread in the perceived distribution of equity returns should raise the equity premium is very intuitive. However, as initially observed by Rothschild and Stiglitz (1971), it is not true in general that risk-averse agents reduce their demand for the risky asset when its return undergoes an increase in risk. Gollier (1995) derives the necessary and sufficient condition for a change in risk to reduce the demand for this risk by all risk-averse investors. There is a wide literature summarized in Gollier (1995) that provides various sufficient conditions. Assuming constant relative risk aversion and lognormal returns, Abel (2002) shows that an increase in the variance of the perceived equity payoffs raises the equity premium.

In the remainder of this section, we examine a purely hypothetical situation of conflicts in beliefs. Our aim is to show that conflicts in beliefs have the potential to have a sizeable effect on the equity premium. Suppose that all agents have the same constant relative risk aversion γ. Consider as a benchmark that all agents believe that the growth rate ω of consumption is lognormally distributed, i.e., that

$$\omega(s) = \exp s \quad \text{and} \quad s \sim N(\mu, \sigma^2).$$

In order to fit the historical U.S. data with an expected growth rate of consumption $E\omega - 1 = 1.8\%$ per year and a standard deviation equaling 3.56% per year, we take $\mu = 0.017$ and $\sigma = 0.035$. Such a specification implies that the equity premium equals $\phi = -1 + \exp(\gamma\sigma^2)$. For the reasonable relative risk aversion $\gamma = 2$, the equity premium is only 0.25% per year, far below the 6% observed equity premium observed during the last century. Hence the equity premium puzzle.

Suppose alternatively that there are two equally-sized groups with heterogeneous beliefs. The first group is pessimistic: its members believe that s is $N(\mu_p, \sigma)$ with $\mu_p < 0.017$. The other group is optimistic, with beliefs on s distributed as $N(\mu_o, \sigma)$. We select μ_p and μ_o in such a way that $0.5(\mu_p + \mu_o)$ equals $\mu = 0.017$ as in the homogeneous economy. Thus, the two groups of agents have wrong beliefs, *but they are right "on average"*. This work thus differs much from Abel (2002) who examines how does a systematic bias in beliefs affect asset prices.

In the calibration presented in Figure 6, we duplicate the degree of heterogeneity of Figures 3 and 5 by assuming that $\mu_p = -0.053$ and $\mu_o = 0.087$ are separated by 4 times the standard deviation. We see that the effect of the divergence of opinions has a sizeable positive effect on the equity premium. For example, for $\gamma = 2$, the equity premium goes from $\phi = 0.25\%$ with the homogeneous beliefs $\mu = 0.017$, whereas it goes up to $\phi = 1.11\%$ with

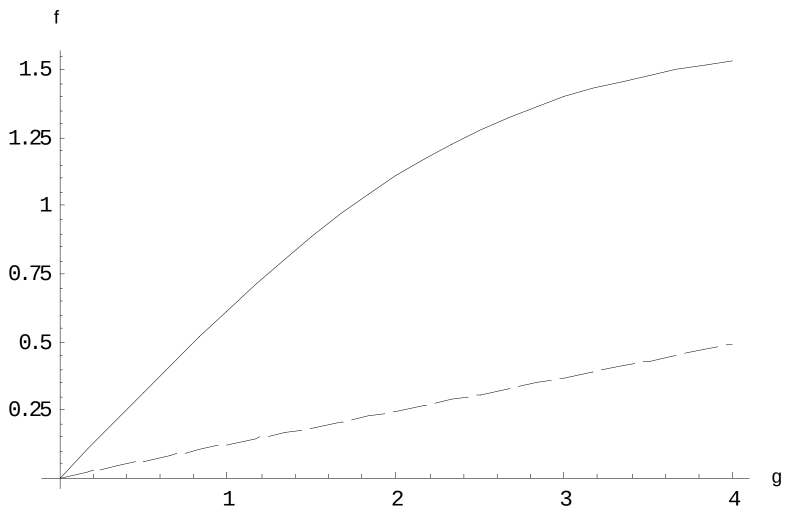

Figure 6: The equity premium ϕ (in % per year) as a function of relative risk aversion γ, with$\mu_p = -5.3\%$ and $\mu_o = 8.7\%$. The dashed curve corresponds to the homogeneous case with $\mu = 1.7\%$.

$\mu_p = -0.053$ and $\mu_o = 0.087$. Thus, the heterogeneity of beliefs multiplies the equity premium by 444%. Of course, to obtain such a result, we took a rather extreme assumption, since optimistic agents believe that the expected growth rate of consumption is 9.2% per year, whereas pessimistic ones believe that expected consumption decreases at a yearly rate of 5.1%. In Figure 7, we show how the equity premium varies with the degree of heterogeneity of beliefs. This relationship is convex. Observe in particular that introducing heterogeneous beliefs has no effect on the equity premium at the margin.

10 The arithmetic aggregation rule

We have seen that the geometric aggregation rule is useful to obtain comparative statics properties, in particular when individual beliefs belong to the family of normal distributions. An alternative benchmark aggregation rule would be to use an arithmetic mean of individual probabilities, state by

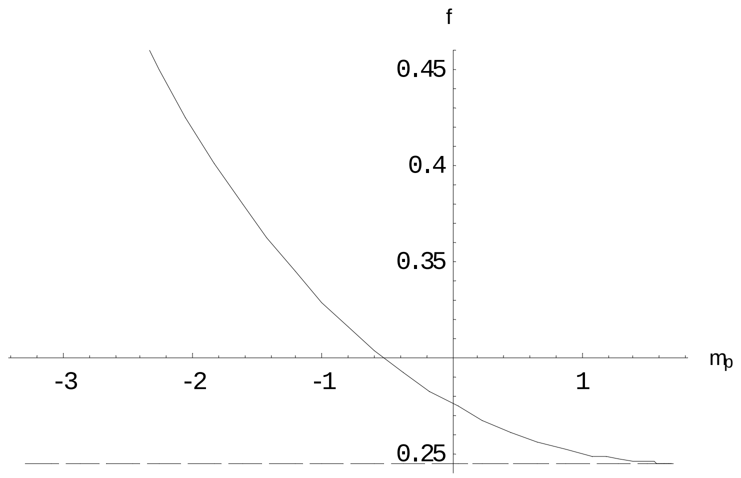

Figure 7: The equity premium ϕ (in % per year) as a function of the expected growth rate μ_p (in % per year) of the pessimistic group with $\gamma = 2$.

state:

$$p^v(P) = k\frac{1}{N}\sum_{\theta=1}^{N}\lambda(\theta)p(\theta), \tag{31}$$

or $d\ln p^v(P) = d\ln\left(N^{-1}\sum_{\theta=1}^{N} q(\theta)\right)$. In Appendix A, we show that this arithmetic aggregation rule is efficient when all agents have the same logarithmic utility function. In such a group, the efficient rule to aggregate heterogeneous beliefs consists in computing for each state the Pareto-weighted mean of the individual subjective probabilities. If two states have the same weighted mean, they should have the same collective probability. If agents do not have a logarithmic utility, this arithmetic aggregation rule is inefficient. The associated error can be estimated by

$$\phi(z, P, dP) = \frac{d\ln v_z(z, P)}{d\ln\left(N^{-1}\sum_{\theta=1}^{N} q(\theta)\right)}. \tag{32}$$

When ϕ is larger than unity, using the arithmetic aggregation rule would underestimate the rate of increase in the collective probability generated by the marginal shift dP. It is easy to check that when dP represents a relative increase in disagreement with respect to P, ϕ is smaller than η. It implies that $\phi > 1$ is a more demanding property than $\eta > 1$. In a previous version of this paper[6], I proved the following proposition using the efficient aggregation rule (28).

Proposition 8 *Suppose that the individual utility functions are identical. The following two conditions are equivalent:*

1. *For any wealth z, any initial distribution of individual probabilities P and any shift dP yielding a relative increase in disagreement, the rate of increase in the collective probability is larger than the rate of increase of the mean individual probabilities: $\phi(z, P, dP) \geq 1$;*

2. *The derivative of absolute risk tolerance with respect to consumption is larger than unity: $T_c^u(c) \geq 1$ for all c.*

Varian (1985) and Ingersoll (1987) proved that 2 implies 1. Comparing this proposition with Proposition 7 clarifies the main difference between our work with the existing literature. Whereas Varian and Ingersoll compared the rate of increase of the collective probability to the rate of increase of the mean individual probabilities, we compare it to the mean rate of increase in individual probabilities. For many applications as the one presented in the previous section with lognormal beliefs, our approach is more useful. Moreover, our condition on preferences to obtain an unambiguous comparative property is much simpler. Notice that

$$T_c^u(c) = \frac{P^u(c)}{A^u(c)} - 1 \text{ with } A^u(c) = -\frac{u''(c)}{u'(c)} \text{ and } P^u(c) = -\frac{u'''(c)}{u''(c)}. \tag{33}$$

A^u and P^u are respectively the degree of absolute risk aversion and absolute prudence. Kimball (1990) shows that absolute prudence is useful to measure the impact of risk on the marginal value of wealth. Namely, he shows that the effect of risk on the marginal value of wealth is equivalent to a sure reduction of wealth that is approximately proportional to the product of the variance of the risk by P^u. Using equation (33), the derivative of absolute

[6]See Gollier (2003).

risk tolerance is larger than unity if and only if absolute prudence is larger than twice the absolute risk aversion:

$$T_c^u(c) \geq 1 \Leftrightarrow P^u(c) \geq 2A^u(c). \tag{34}$$

There is a simple intuition to Proposition 8. It states that, everything else unchanged, the group should devote more effort to finance aggregate consumption in states with more disagreement if P^u is larger than $2A^u$. The paradigm of the veil of ignorance is useful for this intuition, using Proposition 6. Under the veil of ignorance, the cake sharing problem (4) is equivalent to an Arrow-Debreu portfolio problem. More disagreement in the cake sharing problem can be reinterpreted as more risk in the portfolio problem, which has two conflicting effects on the marginal value of aggregate wealth v_z. The first effect is a precautionary effect. The increase in risk has an effect on v_z that is equivalent to a sure reduction of aggregate wealth which is approximately proportional to absolute prudence. But this does not take into account of the fact that the group does rebalance consumption towards those who have a larger probability. This endogenous negative correlation between the weighted probability $q(\theta)$ and individual consumption is favorable to the expected consumption $E\lambda(\widetilde{\theta})p(s,\widetilde{\theta})c(z,s,\widetilde{\theta})$. Under the veil of ignorance, this makes the representative agent implicitly wealthier, thereby reducing the marginal value of wealth. This wealth effect is proportional to the rate at which marginal utility decreases with consumption. It is thus proportional to A^u. Globally, more disagreement raises the marginal value of wealth if the precautionary effect dominates the wealth effect, or if absolute prudence is sufficiently larger than absolute risk aversion. This provides an intuition to condition $P^u \geq 2A^u$, or $T_c^u \geq 1$.[7]

The assumption that agents have decreasing absolute risk aversion is a widely accepted hypothesis in our profession. The plausibility of condition $T_c^u \geq 1$ is more questionable. In fact, most specialists in the field believe that T_c^u is *smaller* than unity. The argument goes as follows. In the absence of any direct estimate of the sensitivity of absolute risk tolerance to changes in wealth, we usually consider the CRRA specification $u(c) = c^{1-\gamma}/(1-\gamma)$ for which $T^u(c) = c/\gamma$. It implies that $T_c^u \geq 1$ if and only if γ is smaller than

[7]In the Arrow-Debreu portfolio context, Gollier (2002) shows that condition $P^u \geq 2A^u$ is necessary and sufficient for a mean-preseving spread in the distribution of state price per unit of probability to raise the marginal value of wealth.

unity. Relying on asset pricing data and the equity premium puzzle, one must conclude that relative risk aversion must be much larger than unity. Therefore, Proposition 8 should be interpreted by considering their contraposition with $T_c^u(c) \leq 1$!

11 Conclusion

Our aim in this paper was to characterize the beliefs that should be used for collective decision making when individuals differ about their expectations. To examine this question, we assumed that agents can share risk efficiently, thereby relying on techniques borrowed from the theory of finance. The basic ingredient behind our results is that, in aggregating individual beliefs, one should favor the beliefs of agents that bear a larger share of the risk. However, the allocation of risk in the economy is endogenous and it depends upon individual beliefs. Therefore, efficient risk allocations are more difficult to characterize under expectations disagreement. For example, it is not necessarily efficient to wash out diversifiable risks in that case. It may be efficient for agents to gamble against each others in spite of their risk aversion. Horse-track betting is Pareto-improving when agents have different beliefs about the chances of the competing horses.

In an Arrow-Debreu framework, the risk exposure of an individual is a local concept that is measured by differences in state consumption levels across states. As is well-known, the socially efficient local risk exposure for an agent is proportional to his local degree of absolute risk tolerance which measures the rate at which marginal utility decreases with consumption. We showed that this result remains true with heterogeneous beliefs. The key property of the aggregation of beliefs is that an increase in the subjective state probability of agent θ should raise the collective probability also proportionally to agent θ's degree of absolute risk tolerance. If an agent bears a percentage share x of the collective risk, a one percent increase in his subjective probability should raise the collective probability by x percents. This result has several important consequences.

First, it implies that the socially efficient collective probability distribution depends upon the aggregate wealth level of the group. This is because the aggregate wealth level affects the way risks should be allocated in the group. However, when agents have the same HARA utility function, changes in aggregate wealth has no effect on the allocation of risks. This implies that the collective probability distribution is independent of wealth in that case. We showed that the identically-sloped HARA case is the *only* case in which such separability property between beliefs and utility holds. In all other cases, the representative agent has a state-dependent utility function.

Second, we derived various results that are useful to understand the effect of the divergence of opinions on the shape of the collective probability distribution. To do this, we defined the concept of increasing relative disagreement. In short, there is more relative disagreement about state s' than about state s if the individual subjective probabilities are more dispersed in state s' than in state s. We showed that, with such a shift in the distribution of individual probabilities, the rate of increase of the collective probability is larger than the mean rate of increase of individual probabilities if and only if absolute risk aversion is decreasing. It must be stressed that this result is purely local. It does not provide a global view about how the beliefs of the representative agent are affected by the heterogeneity of beliefs.

The last step is to link the structure of disagreement at the global level to the global properties of the collective probability distribution. When most disagreements are concentrated in the tails of the distribution, the collective distribution function is dominated by the average individual probability distribution in the sense of second-order stochastic dominance. This tends to raise the equity premium. We showed in a simple numerical example that the heterogeneity of individual beliefs may have a sizeable effect on the equity premium.

The critical assumption of this model is that the group can allocate risk efficiently. This assumption is difficult to test. For example, the efficient coverage of earthquake coverage in various regions can be interpreted in two ways. The optimistic view is that homeowners are more pessimistic than insurers about the risk, which implies that the low insurance coverage is socially efficient. But alternatively, it could be interpreted as a proof that markets are incomplete. A similar problem arises to explain the insurance crisis after 9/11/01, or about the difficulty to share the risk related to global warming on an international basis. A possible extension of this work would be to consider an economy with incomplete markets.

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Appendix A: The case of ISHARA preferences

In this appendix, we examine the special case of ISHARA preferences (21) for which we know from proposition 4 that R is independent of z. It implies that $v_z(z,P)$ is separable into a product $p^v(P)h'(z)$. Moreover, ISHARA preferences (21) yield an analytical solution for the aggregation problem. Indeed, in this particular case, the first-order condition to state-dependent the Pareto program (4) implies that

$$c(z,P,\theta) - a(\theta) = k\left[\lambda(\theta)p(\theta)\right]^{1/\gamma}.$$

Since $T^u(c,\theta) = (c - a(\theta))/\gamma$, property (14) can be rewritten in the ISHARA case as

$$R(z,P,\theta) = \frac{\left[\lambda(\theta)p(\theta)\right]^{1/\gamma}}{NE\left[\lambda(\widetilde{\theta})p(\widetilde{\theta})\right]^{1/\gamma}}, \tag{35}$$

where $Ef(\widetilde{\theta}) = N^{-1}\sum_{\theta=1}^{N} f(\theta)$. The definition of R applied to the ISHARA case implies that

$$R(z,P,\theta) = \frac{p(\theta)p^v_\theta(P)}{p^v(P)}, \tag{36}$$

where $p^v_\theta = \partial p^v/\partial p(\theta)$. Combining (35) and (36) yields

$$\frac{p^v_\theta(P)}{p^v(P)} = \frac{\lambda(\theta)^{1/\gamma}p(\theta)^{-1+1/\gamma}}{NE\left[\lambda(\widetilde{\theta})p(\widetilde{\theta})\right]^{1/\gamma}} \tag{37}$$

for $\theta = 1, ..., N$. The solution to this system of partial differential equations has the following form:

$$p^v(P) = C \quad \left[E\left[\lambda(\widetilde{\theta})p(\widetilde{\theta})\right]^{1/\gamma}\right]^{\gamma}, \tag{38}$$

where C is a constant. In order for p^v to be a probability distribution, we need to select the particular solution with

$$p^v(P(s)) = \frac{\left[E_{\widetilde{\theta}}\left[\lambda(\widetilde{\theta})p(s,\widetilde{\theta})\right]^{1/\gamma}\right]^{\gamma}}{\sum_{t=1}^{S}\left[E_{\widetilde{\theta}}\left[\lambda(\widetilde{\theta})p(t,\widetilde{\theta})\right]^{1/\gamma}\right]^{\gamma}}. \tag{39}$$

Calvet, Grandmont and Lemaire (2001) obtained the same solution. Rubinstein (1974) derives it in the special cases $\gamma = 1$ and $\gamma = +\infty$.[8] Chapman and Polkovnichenko (2006) derived this result in the special case of CRRA. Thus, in the ISHARA case, we can directly compute the socially efficient probability distribution of risk as a function of individual beliefs p, the Pareto weights λ, and the concavity coefficient γ. Two special cases are worthy to examine. Consider first the case with γ tending to zero. This corresponds to risk-neutral preferences above a minimum level of subsistence. Under this specification, condition (39) is rewritten as

$$p^v(P(s)) = p^n(P(s)) =_{def} \frac{\max_{\theta\in\Theta}\lambda(\theta)p(s,\theta)}{\sum_{t=1}^{S}\max_{\theta\in\Theta}\lambda(\theta)p(t,\theta)} \quad \text{for all } s. \text{ (risk-neutral case)} \tag{40}$$

With risk-neutral preferences, the efficient allocation produces a flip-flop strategy where the cake in state s is entirely consumed by the agent with the largest Pareto-weighted probability associated to that state. It implies that the group will use a state probability p^n proportional to it to determine its attitude toward risk ex ante.

In the case of logarithmic preferences ($\gamma = 1$), the denominator in (39) equals $E\lambda(\widetilde{\theta})$ since

$$\sum_{t=1}^{S} E_{\widetilde{\theta}}\lambda(\widetilde{\theta})p(t,\widetilde{\theta}) = E_{\widetilde{\theta}}\left[\lambda(\widetilde{\theta})\sum_{t=1}^{S}p(t,\widetilde{\theta})\right] = E\lambda(\widetilde{\theta}) = 1.$$

It implies that

$$p^v(P(s)) = p^{\ln}(P(s)) =_{def} E\lambda(\widetilde{\theta})p(s,\widetilde{\theta}) \quad \text{for all } s. \text{ (logarithmic case)}$$

With these Bernoullian preferences, the efficient probability that should be associated to any state s is just the weighted mean $p^{\ln}(s)$ of the individual subjective probabilities of that state s. This is the limit case $T_c^u \equiv 1$ of the result presented in Proposition 8.

Appendix B: Increasing disagreement in the large

[8] The CARA case$\gamma = +\infty$ is described in section 7.

In this Appendix, we extend our definition of increasing disagreement in the small to non-marginal changes in individual probabilities. We examine non marginal changes in distribution by comparing two distributions P_0 and P_1. Our definition of an increase in disagreement "in the large" is as follows.

Definition 2 *Consider a specific Pareto-weight vector* $(\lambda(1), ..., \lambda(N))$. *We say that* P_1 *yields more disagreement than* P_0 *if* $q_0(\widetilde{\theta}) = \lambda(\widetilde{\theta})p_0(\widetilde{\theta})$ *and* $\ln q_1(\widetilde{\theta}) - \ln q_0(\widetilde{\theta})$ *are comonotone: for all* (θ, θ') :

$$[q_0(\theta') - q_0(\theta)] \left[\ln \frac{q_1(\theta')}{q_0(\theta')} - \ln \frac{q_1(\theta)}{q_0(\theta)} \right] \geq 0.$$

If q_0 is increasing in θ, this is equivalent to require that (P_0, P_1) satisfies the Monotone Likelihood Ratio (MLR) property that $p_1(\theta)/p_0(\theta)$ be increasing in θ. Because agents with a larger lnq under P_0 get a larger increase in log probabilities under P_1, it implies that the distribution of log probabilities under P_1 is a spread of the individual log probabilities under P_0. Thereby, it amplifies the dispersion of $q(\widetilde{\theta})$.

It is useful to decompose any shift in distribution from P_0 to P_1 as a sequence of infinitesimal changes in probabilities $dP(\tau) = (dp(\tau, 1), ..., dp(\tau, N))$ indexed by τ going from 0 to 1 with

$$P(t) = P_0 + \int_0^t dP(\tau) \geq 0 \text{ and } P(1) = P_1.$$

Among the various ways to do this, we are interested in the paths $P(.)$ that preserve the property of increasing disagreement for each infinitesimal change $dP(\tau)$ in the vector of individual probabilities. The following Lemma proves that such paths exist.

Lemma 1 *If* P_1 *exhibits more disagreement than* P_0, *there exists a path* $P(.)$ *linking* P_0 *to* P_1 *in which each increment* $dP(\tau)$ *yields an increase in disagreement.*

Proof: We check that $P(t) = P_1^t P_0^{1-t} = P_0 \exp[t \ln P_1/P_0]$ satisfies this property. Define $q(t, \theta) = \lambda(\theta)p(t, \theta) = \lambda(\theta)p_1(\theta)^t p_0(\theta)^{1-t}$. It implies that

$$d \ln q(t, \theta) = \ln \frac{p_1(\theta)}{p_0(\theta)} dt, \tag{41}$$

which is independent of t. Without loss of generality, suppose that P_0 is such that $q_0(1) \leq q_0(2) \leq ... \leq q_0(N)$. Because P_1 exhibits more disagreement than P_0, it must be that $p_1(\theta)/p_0(\theta) = q_1(\theta)/q_0(\theta)$ be increasing in θ. Combining this with equation (41) implies that the right bracketed term in (27) is positive if $\theta' > \theta$. It remains to prove that $\lambda(\theta)p_1(\theta)^t p_0(\theta)^{1-t}$ is increasing in θ. This is immediate from the observation that

$$\lambda(\theta)p_1(\theta)^t p_0(\theta)^{1-t} = q_1(\theta)^t q_0(\theta)^{1-t} = q_0(\theta)\left[\frac{q_1(\theta)}{q_0(\theta)}\right]^t$$

is the product of two positive increasing functions of θ. Notice that this implies that $\lambda(\theta)p(t,\theta)$ increases with θ at a rate that increases with t,or that $\lambda(\theta)p(t,\theta)$ is logsupermodular. ■

This Lemma is useful because it allows us to focus on marginal changes in distribution. Any result holding for increasing disagreement in the small can be extended to increases in disagreement in the large. For example, because a sequence of increases in risk is an increase in risk, Proposition 6 implies that P_1 is riskier than P_0 in the sense of Rothschild-Stiglitz if P_1 exhibits more disagreement than P_0 and $Eq_1(\widetilde{\theta}) = Eq_0(\widetilde{\theta})$.

Appendix C: Proof of Proposition 6

We check that for any concave function h, $N^{-1}\sum_{\theta=1}^{N} h(\ln q(\theta))$ is reduced by the marginal shift dP. Suppose without loss of generality that q is increasing in θ. Because dP is an increase in disagreement, we have that $d\ln q(\theta)$ is increasing in θ. By the covariance rule, it implies that

$$\begin{aligned} N^{-1}d\sum_{\theta=1}^{N} h(\ln q(\theta)) &= N^{-1}\sum_{\theta=1}^{N} h'(\ln q(\theta))d\ln q(\theta) \\ &\leq N^{-1}\sum_{\theta=1}^{N} h'(\ln q(\theta))\left[N^{-1}\sum_{\theta'=1}^{N} d\ln q(\theta')\right] = 0.(42) \end{aligned}$$

The last equality comes from the assumption that dP preserves the mean of $\ln q(\theta)$. This proves the sufficiency part of the proposition. The proof of necessity is by contradiction. Suppose that $d\ln q(\theta)$ is decreasing in a neighborhood of some θ_0. Then, inequality (42) is reversed for any function h that is linear outside this neighborhood and concave inside it, thereby contradicting the condition that dP yields a Rothschild-Stiglitz spread of $(\ln q(1), ..., \ln q(N))$. ■