Wealth Inequality and Asset Pricing

Christian Gollier
GREMAQ and IDEI, Université de Toulouse [1]
Email: Gollier @ cict.fr

May 2000

[1] I thank Pierre-André Chiappori, John Geneakoplos, Giovanni Immordino, Miles Kimball, Andrew Lo, Ben Polak, Edward Schlee, Philippe Weil and three anonymous referees for helpful discussions.

Abstract

In an Arrow-Debreu exchange economy with identical agents except for their initial endowment , we examine how wealth inequality affects the equilibrium level of the equity premium and the risk-free rate. We first show that wealth inequality raises the equity premium if and only if the inverse of absolute risk aversion is concave in wealth. We then show that the equilibrium risk-free rate is reduced by wealth inequality if the inverse of the coefficient of absolute prudence is concave. We also prove that the combination of a small uninsurable background risk with wealth inequality biases asset pricing towards a larger equity premium and a smaller risk-free rate.

Keywords: wealth inequality, equity premium, risk free rate, prudence, background risk.

1 Introduction

The model of the representative consumer à la Lucas (1978) failed to explain observed security prices. More specifically, as first stated by Mehra and Prescott (1985), the model predicts an equity premium that is too low, and a risk free rate that is too high, given the observed low variability of consumption growth. A recent strand of the literature with papers by Constantinides and Duffie (1996), Heaton and Lucas (1996) and Krusell and Smith (1997) for example, explores whether incomplete asset markets and heterogeneity among consumers can explain these puzzles. Constantinides and Duffie (1996) in particular conclude that it is, at least in principle, possible to generate larger equity premia when such ingredients are introduced. The aim of this paper is not to provide another explanation of the puzzles, but rather to better understand the relationship between asset prices, the heterogeneity of consumers' wealth, and the market incompleteness.

When markets are complete, heterogeneous consumers are able to equalize their marginal rates of substitution across states and across time. As shown by Constantinides (1982), this implies that the equilibrium of the heterogeneous-consumer economy can be duplicated by the equilibrium in a Lucas economy with a representative agent. This means that taking into account of the heterogeneity in the population alone cannot explain why the representative-agent model in rejected (Hansen and Singleton (1982)). This does not mean, however, that the distribution of wealth in the population, for example, has no effect on asset prices in general. In order to understand this, consider the departure from an egalitarian distribution of wealth by introducing a mean-preserving transfer of endowment. Under decreasing absolute risk aversion, poorer agents will be more risk-averse, whereas wealthier people will be less risk-averse. There is no reason a priori for that the group's attitude towards risk be unaffected by this transfer. We first show that if absolute risk tolerance, i.e., the inverse of the Arrow-Pratt coefficient of absolute risk aversion, is concave, then the risk aversion of the agent with the mean wealth will be smaller than the risk aversion of the representative agent. Thus, if an outside observer overlooks wealth inequality, he consequently underestimates the aversion to risk of Society. This leads to an underestimation of the equity premium. The opposite effect holds if the absolute risk tolerance is convex.

A side product of this analysis is that the linearity of absolute risk tolerance is necessary and sufficient for the absence of any effect of wealth inequality on asset prices. It is noteworthy that the existing literature on the equity premium puzzle systematically uses a specification of the Lucas model with constant relative risk aversion, a special case of linear risk tolerance. When combining this standard assumption with complete insurance markets, the introduction of wealth inequality in the modeling of the economy does not bring anything to the solution of the puzzles. From this standpoint, this paper explores the possibility that the elimination of one of these two assumptions (CRRA versus complete markets) can help explain asset pricing anomalies.

We begin with the removal of the assumption that relative risk aversion is constant. We claim that there is no clear argument in favor of such a specification. Quite the contrary, the constancy of relative risk aversion generates several properties of optimal decisions that are at odd with the observation. For example, it implies that the relative share of wealth invested in risky assets is constant with wealth. It also implies that the age of the investor does not affect the optimal portfolio composition, i.e., that myopia is optimal (Mossin (1968)). Finally, it generates a constant marginal propensity to consume under certainty. These properties are contradicted by all empirical studies on household portfolios and savings. While various arguments that are not based on preferences may be given to explain the discrepancy, this evidence is suggestive enough to open the door for the consideration of other utility functions. We give more insights about our position on this subject in section 4. Notice also that Kandel and Stambaugh (1991), using a thought experiment on the relationship between the size of a bet and the associated risk premium, came to the conclusion that there is no single constant relative risk aversion that is compatible with the risk premium for small and large sizes at the same time. In section 5, we show by how much we can increase the equity premium by considering a concave absolute risk tolerance. Using a simple approximation of the distribution of wealth in the U.S. together with a sensible utility function, we show that wealth inequality can double the equity premium.

The existing literature did not follow this road. Rather, they maintained the CRRA assumption, but dropped the assumption that markets are complete, as in Constandinides and Duffie (1996) and Heaton and Lucas (1996). All tests using disaggregated consumption data, as in Townsend (1994) and Attanasio and Davis (1993), clearly reject the hypothesis that risks are efficiently shared. If they are not, this is either because markets are incomplete, or because investors face transaction costs and various constraints on these markets. In section 7, as in Weil (1992) and many others, we take the view that future labor incomes cannot be insured. As explained above, wealth inequality does not matter for asset prices in the CRRA case when the flow of uncertain labour incomes can be traded. We show that wealth inequality has a positive effect on the equity premium when labour incomes cannot be insured. However, the size of the impact is extremely small. We conclude that departing from the linear risk tolerance case is necessary to take wealth inequality as a serious element to explain the puzzle.

In this paper, we also examine how wealth inequality affects the equilibrium risk-free rate. This problem is more complex, since the equilibrium risk free rate is affected not only by the elasticity of intertemporal substitution, but also by the degree of prudence. How does wealth inequality affect the way individual elasticities of substitution and degrees of prudence must be aggregated is the relevant question here. It is shown that this rate is reduced by wealth inequality if the inverse of absolute prudence is concave in wealth. This can explain why a general equilibrium model that is calibrated on the basis of the average consumption growth, not taking into account of the unequal distribution, will overpredict the risk-free rate.

2 Characterization of the representative agent

The economy is made of $n > 1$ individuals who all have the same attitude towards risk and consumption smoothing over time. It is characterized by the von Neumann-Morgenstern utility function u on consumption at each period. This function is increasing and concave. There are two dates indexed by $t = 0$ or $t = 1$. There is a single good that is perishable. At $t = 0$, there is uncertainty about the state of the world that will prevail at $t = 1$. The uncertainty takes the form of S states of the world indexed by $s = 1, ..., S$, with a probability p_s of occurrence of state s. Each agent i, $i = 1, ..., n$, is endowed with a ω_{i0} units of the consumption good at $t = 0$, and with a bundle of contingent claims $(\omega_{i1}, ..., \omega_{is}, ..., \omega_{iS})$. There is an aggregate uncertainty since

$$z_s =_{\text{def}} \sum_{i=1}^{n} \frac{1}{n} \omega_{is}$$

is not constant in s, i.e., the average endowment z_s is random.

We assume that there is a frictionless, competitive market for contingent claims that takes place at $t = 0$. Let π_0 and π_s be the price to be paid at $t = 0$ for the delivery of 1 unit of the good respectively at $t = 0$ and at $t = 1$, conditional on state s. The standard saving-portfolio problem of agent i is written as follows:

$$\max_{c_{i.}} \quad u(c_{i0}) + \beta \sum_{s=1}^{S} p_s u(c_{is}) \tag{1}$$

$$\text{s.t. } \pi_0(c_{i0} - \omega_{i0}) + \sum_{s=1}^{S} \pi_s(c_{is} - \omega_{is}) = 0, \tag{2}$$

where β is the discount factor. Let us denote $q_0 = 1/\beta$ and $q_s = p_s$ for $s = 1, ..., S$. The above problem can then be rewritten as

$$\max_{c_{i.}} \quad \sum_{s=0}^{S} q_s u(c_{is}) \tag{3}$$

$$\text{s.t. } \sum_{s=0}^{S} \pi_s(c_{is} - \omega_{is}) = 0, \tag{4}$$

By operating this rewriting, we transform a two-period saving-portfolio decision problem into a static problem of investment under uncertainty. This is standard within the expected utility framework with a time-separable utility function. The elimination of the saving decision in problem (3) is an illustration of the inability of the time-separable-utility model to disentangle risk aversion from the attitude towards the intertemporal substitutability of consumption.

The competitive equilibrium satisfies the following set of necessary and sufficient conditions:

$$q_s u'(c_{is}) = \lambda_i \pi_s, \quad i = 1, ..., n, \quad s = 0, ..., S, \tag{5}$$

and

$$\sum_{i=1}^{n} \frac{1}{n} c_{is} = z_s, \quad s = 0, ..., S. \tag{6}$$

Conditions (5) are the first-order conditions of the maximization problem of agent i, with λ_i being the Lagrangian multiplier associated to her budget constraint. Condition (6) is the market-clearing condition associated to the market for state-claims s. This condition for $s = 0$ means that the good is perishable, i.e., it cannot be transferred to the next period.

A well-known property of any Pareto-efficient allocation in this model is the so-called mutuality principle: if $z_s = z_{s'}$, then $c_{is} = c_{is'}$ for all i. This is another way to say that all diversifiable risks have been washed out at equilibrium. It implies in particular that the competitive equilibrium can be expressed by a price kernel $\pi_s/q_s = \pi(z_s)$ and by a set of functions $c_i(.)$ such that $c_{is} = c_i(z_s)$, for all i and s. A better characterization can then be obtained by fully differentiating condition (5). Eliminating λ_i, it yields

$$\frac{dc_i(z)}{dz} = \frac{-\pi'(z)}{\pi(z)} T(c_i(z)), \tag{7}$$

where $T(c) = -u'(c)/u''(c)$ is the Arrow-Pratt degree of absolute risk tolerance. Using the market-clearing condition (6), we obtain

$$\frac{-\pi'(z)}{\pi(z)} = n \left[\sum_{i=1}^{n} T(c_i(z)) \right]^{-1}, \tag{8}$$

and

$$\frac{dc_i(z)}{dz} = \frac{T(c_i(z))}{\sum_{j=1}^{n} \frac{1}{n} T(c_j(z))}. \tag{9}$$

In words, the risk borne by individual i at equilibrium is proportional to his absolute risk tolerance. This result is well-known and has first been obtained by Wilson (1968).[1]

Since Lucas (1978), it is standard to estimate the equilibrium price of assets by using the model presented here above, but with n identical agents

[1] See also Raviv (1979). Leland (1980) provides conditions for the individual payoff functions c_i to be concave or convex in z.

facing the same risk $\tilde{z}$ on their initial bundle. If this homogenous economy duplicates the equilibrium contingent prices obtained in an economy with heterogenous agents facing the same uncertainty, we say that these identical agents are "representative" of this heterogenous economy. Constantinides (1982) showed that if markets are complete, there exists a representative agent to any heterogenous economy. The representative agent is characterized by a utility function $\hat{u}$ such that

$$\hat{u}'(z) = \pi(z) \tag{10}$$

for all z, where $\pi(.)$ is the price kernel of the heterogenous economy. Function $\hat{u}$ represents the attitude towards risk and time of the heterogenous economy as a whole.

In general, there is no simple way to characterize the attitude towards risk of the representative agent, except in two cases. First, trivially, if there is no heterogeneity in the economy, the equilibrium will be symmetric ($c_i(z) = z$). It implies that $\hat{u} \equiv u$ in that case. Second, let u belong to the set of utility functions with an harmonic absolute risk aversion (HARA):

$$u'(c) = (a + \frac{c}{\gamma})^{-\gamma}, \tag{11}$$

for some constants a and γ. Solving condition (5) yields

$$c_i(z) = -a\gamma + \gamma(\pi(z))^{-\frac{1}{\gamma}}\lambda_i^{-\frac{1}{\gamma}}.$$

Adding up this equality with respect to $i = 1, ..., n$ and using the market-clearing condition (6) yields

$$z = -a\gamma + \gamma(\pi(z))^{-\frac{1}{\gamma}} \sum_{i=1}^{n} \frac{1}{n}\lambda_i^{-\frac{1}{\gamma}},$$

or,

$$\begin{aligned} \hat{u}'(z) &= \pi(z) \\ &= \left[\textstyle\sum_{i=1}^{n} \frac{1}{n}\lambda_i^{-\frac{1}{\gamma}}\right]^{\gamma} (a + \frac{z}{\gamma})^{-\gamma} \\ &= Ku'(z) \end{aligned} \tag{12}$$

Thus, when u is HARA, the representative agent has the same attitude towards risk — and the same elasticity to intertemporal substitution — as the original agent in the economy, even if the distribution of wealth is unequal. In particular, wealth inequality will have no effect on asset pricing in this economy. It will be a consequence of our analysis that HARA functions are the only ones to exhibit such a property.

Our objective is to determine the effect of wealth inequality on asset pricing when the utility function is not HARA. An asset is defined by its payoff vector $(X_s)_{s=0,1,...,S}$. Its equilibrium price equals $\sum_s \pi_s X_s = \sum_s q_s X_s \hat{u}'(z_s)$. We will be more specifically interested in the pricing of two assets: an asset whose return is perfectly correlated with the aggregate risk ("equity"), and the risk-free asset ("bond"). The payoff vector of equity is $X^e = (0, z_1, ..., z_s, ..., z_S)$, whereas the payoff vector of a bond is $X^b = (0, 1, ..., 1, ..., 1)$.

3 The equity premium

We are interested here in the relative price of equity with respect to bonds. It is given by

$$\Pi = \frac{\sum_{s=1}^{S} p_s z_s \hat{u}'(z_s)}{\sum_{s=1}^{S} p_s \hat{u}'(z_s)}. \tag{13}$$

The following two lemmas are useful to determine the effect of wealth inequality on the relative price of equity. The first Lemma is already in Weil (1992).

Lemma 1 *Wealth inequality reduces the relative price* Π *of equity with respect to bonds if and only if* $\hat{u}$ *is more risk-averse than* u*.*

Proof: See the Appendix.

Lemma 1 states that the price of equity is reduced by wealth inequality if the representative agent is more risk-averse than the original agent. An increase in risk aversion of the representative agent makes him more reluctant to retain risk in his portfolio. In order to induce him to retain his share of the aggregate risk, i.e. to induce him not to sell his stocks, the relative price of stocks with respect to bonds must be reduced.

Lemma 2 *The representative agent is more (resp. less) averse to risk than the original consumers, i.e. $\hat{u}$ is more risk-averse than u, if and only if the absolute risk tolerance of u is concave (resp. convex).*

Proof: Let $\hat{T}(y) = -\hat{u}'(y)/\hat{u}''(y)$ denote the absolute risk tolerance of the representative agent at y, the differentiation of condition (10) together with condition (8) implies that

$$\hat{T}(z) = -\frac{\hat{u}'(z)}{\hat{u}''(z)} = -\frac{\pi(z)}{\pi'(z)} = \sum_{i=1}^{n} \frac{1}{n} T(c_i(z)). \tag{14}$$

Since z is the mean of the $(c_i(z))_{i=1,...n}$, using Jensen's inequality on the above condition implies that $\hat{T}(z) \leq T(z)$ if T is concave. Since this is true for all z, it directly yields the sufficiency of the concavity of T. The necessity is based on the same argument, but with potential consumption levels that are concentrated on an interval where T is not concave. ■

This proof is based on the fact that the representative agent's absolute tolerance to risk is just the average of the original agents' absolute tolerance to risk. The combination of the two lemmas yields the following result:

Proposition 1 *If absolute risk tolerance is concave (resp. convex), then the equilibrium price of equity in an unequal economy is smaller (resp. larger) than the equilibrium price in the egalitarian one, with the same aggregate uncertainty.*

This proposition generalizes what we already know when absolute risk tolerance is linear, which corresponds to HARA utility functions for which wealth inequality has no effect on asset pricing.

To obtain this result, we don't need to make any assumption on whether risk aversion must be increasing or decreasing. But under increasing absolute risk tolerance, the intuition is as follows: introducing wealth inequality in the economy alters the aggregate demand for equity in two ways. First, poorer agents are more risk averse. This reduces their demand for equity. Second, wealthier people have a larger demand for it, since they are less risk averse. Under linear risk tolerance, we know that the demand for equity is linear in wealth. It implies that the two effects exactly compensate each other, so that the aggregate demand at the original equilibrium price is unaffected. Some intuition can be obtained by observing that,at least for a small aggregate risk, the demand for equity is linear with respect to absolute risk tolerance. Combining this with the fact the absolute risk tolerance of the representative agent is the mean of the actual agents' absolute risk tolerance implies the result. Although this technique cannot be extended to larger risks, Proposition 1 indicates that the same result holds in general.

A consequence of this analysis is that *the equity premium is increased due to wealth inequality if T is concave*, whereas it is decreased if T is convex. Following Weil (1992), suppose that an outside observer is willing to confront the equity premium obtained from the model above with the observed equity premium data. Suppose that the observer believes that an egalitarian allocation is implemented in the economy. This will lead to an underprediction (resp. overprediction) of the equity premium if T is concave (resp. convex). In order to help solving the equity premium puzzle, we thus need T to be concave.

4 The concavity of absolute risk tolerance

What do we know about the sensitivity of absolute risk tolerance to changes in wealth? A familiar hypothesis is that absolute risk tolerance be increasing with wealth (DARA), which means that wealthier people invest more in risky assets. This can be checked by considering the problem of the demand for equity:

$$\max_{\alpha} \ Eu(y_0 + \alpha(\tilde{z} - \Pi)).$$

The first-order condition is $E(\tilde{z} - \Pi)u'(y_0 + \alpha(\tilde{z} - \Pi)) = 0$. If the risk on $\tilde{z}$ is small, the deviation of $(\tilde{z} - \Pi)$ from 0 will be small at equilibrium. Using a Taylor approximation for u' in the first-order condition implies that

$$\alpha(y_0) \cong \frac{E(\tilde{x} - \Pi)}{E(\tilde{x} - \Pi)^2} T(y_0).$$

The demand for risky assets is approximately proportional to absolute risk tolerance. Another consequence of this exercise is to show that T is concave if and only if[2] the demand for risky assets is increasing with wealth at a decreasing rate. The data are clearly against this assumption. Vissing-Jorgensen (1999) shows that the poors do not hold any stock at all, whereas the demand for stocks increase rapidly once households are rich enough to participate to the stock market. But this is best explained by the existence of fixed participation costs which Vissing Jorgensen estimates at between 100 and 200 dollars. A better approach is thus to limit the analysis to the subset of households who have a positive demand for stocks. Vissing-Jorgensen, following Guiso, Jappelli and Terlizzese (1996), obtains that the relative share of stocks in total wealth increases with total wealth. This is not possible if T is concave.

We can link the concavity of absolute risk tolerance to other properties of the optimal behavior with respect to risk and time. For example, Gollier and Zeckhauser (1997) consider a model with frictionless complete financial markets. They compare the optimal portfolio of two agents with the same aggregate wealth, but with different time horizons for consuming the value of their portfolio. The effect of the time horizon on the optimal portfolio has long been debated in finance.[3] Gollier and Zeckhauser (1997) show that the concavity of absolute risk tolerance is necessary and sufficient for younger investors to invest *less* in risky assets. Various econometric analysis of the relation between portfolio composition and age are in Guiso et al. (1996), Bertaut and Haliassos (1997) and Poterba and Samwick (1997). All studies report a negative slope of the stock holding-age profile up to around retirement. For example, Guiso et al. (1996) report that the share of risky assets in wealth is lowest for young households and increases by 20 percentage points through the life cycle to reach a maximum at age 61. This suggests that absolute risk tolerance be convex. But these studies define wealth as the cash-on-hand, whereas in Gollier and Zeckhauser, wealth is aggregate wealth, which includes the net present value of future labour incomes. As argued by Jagannathan and Kocherlakota (1996) and Cocco, Gomes and Maenhout (1998), a simple argument for why the youngest of two

[2]This is formally true only for small risks on returns. We have not been able to confirm this property for larger risks.

[3]See Samuelson (1963) for an original discussion.

agents with the same *cash-on-hand* invests more in risky assets comes from the substitutability of labour incomes and the safe asset. As the consumers grows older and the annuity value of their future earnings declines, they will rebalance their portfolio towards safer assets. Notice also that several other arguments than those related directly to risk preferences could potentially explain the negative risk exposure-age profile: informational considerations, education, health status, borrowing constraints,....

Another property of the concavity of absolute risk tolerance is that it is necessary and sufficient for the marginal propensity to consume (MPC) to be increasing with wealth, under certainty (assuming a positive growth of consumption). The data contradict again this assumption. For example, Lusardi (1992) observed that the MPC is substantially higher for poor consumers. A discussion of the very low MPC for the rich is in Carroll (1998). Here also, various alternative effects may be at play to generate a concave consumption function. The most obvious argument comes from the existence of a liquidity constraint. Another one is related to the uncertainty affecting future incomes, as shown by Carroll and Kimball (1996).

The concavity of absolute risk tolerance is related to the convexity of absolute risk aversion. If $A(z) = T(z)^{-1}$ denotes absolute risk aversion, $T''(z) \leq 0$ is equivalent to

$$A''(z) \geq 2\frac{[A'(z)]^2}{A(z)}. \tag{15}$$

Thus, the concavity of T is stronger than the convexity of absolute risk aversion. Gollier and Pratt (1996) argued that $A'' > 0$ should be regarded as a natural assumption, "since it means that the wealthier an agent is, the smaller the reduction in risk premium following an increase in wealth". Also, under the familiar condition of decreasing A, the concavity of A is not plausible, since a function cannot be positive, decreasing and concave everywhere. If we use the index of absolute prudence $P(z) = -u'''(z)/u''(z)$, condition (15) can be rewritten as

$$\frac{P'(z)}{P(z)} \leq \frac{A'(z)}{A(z)}, \tag{16}$$

i.e., prudence must decrease at a faster rate than risk aversion. Kimball (1993) provides several arguments in favor of the assumption of standardness, which means that both A and P are decreasing. From this characterization of standardness and from inequality (16), we directly obtain the following result, which is originally due to Hennessy and Lapan (1998):

Proposition 2 *If absolute risk tolerance is increasing and concave, then risk aversion is standard.*

Notice that neither the convexity of absolute risk aversion nor standardness are necessary for the concavity of T. This means that the reasonableness of these two assumptions does not say much about the reasonableness of the concavity of absolute risk tolerance. The (weak) conclusion of this section is that the evidence in favor of the concavity or convexity of T while suggestive, cannot be taken as conclusive.

5 Numerical estimations

At this stage, it is interesting to estimate by how much the equity premium is affected by wealth inequality. To make things simple, let us consider the following very simple macroeconomic uncertainty. There are two states of the world at date 1. In the recession state 1, the consumption per capita is reduced by 1% ($z_1 = 0.99$), whereas it is increased by 6.32% ($z_2 = 1.0632$) in state 2. The probability of the recession state is 0.61756. The mean and the variance of the growth rate of this economy are the same as those of the U.S. economy for the period 1889-1978, as reported by Kocherlakota (1996).

We follow Guiso and Paiella (2000) for the specification of the utility function. We assume that absolute risk tolerance is such that

$$T(z) = \frac{z^b}{r}, \quad \text{or} \quad u'(z) = \exp\left[-r\frac{z^{1-b}}{1-b}\right] \tag{17}$$

where $r > 0$ measures the degree of relative risk aversion evaluated at consumption level $z = 1$. In the following simulations, we took $r = 2$, which is considered as a reasonable level of risk aversion by many economists. As observed by Guiso and Paiella (2000), this specification generalizes several special cases, as the CRRA case ($b = 1$). Notice moreover that absolute risk tolerance is concave when $b \in \,]0, 1[$, and it is convex when b is larger than unity.

We first consider a very simple distribution of wealth in the economy, with two equally weighted social classes, as in Heaton and Lucas (1996). The poors are endowed with a share $(1-h)$ of the GDP per capita in each state, whereas the riches get a share $(1+h)$ of it. Parameter h is at the same time the standard deviation and the coefficient of variation of the wealth distribution. Using Mathematica©, we computed the equity premium of this economy for h going from 0 to 0.5. When $b = 0.5$ (concave absolute risk tolerance), the equity premium equals 0.24% in the egalitarian economy. It goes up to 0.28% when the coefficient of variation h is increased from zero to 0.9. When $b = 2$ (convex absolute risk tolerance), the equity premium goes from 0.24% in the egalitarian economy down to 0.13% when the coefficient of variation of the wealth distribution is raised to $h = 0.9$.

The U.S. economy is much more unequal than what we assumed in the above simulations. Diaz-Gimenez, Quadrini and Rios-Rull (1996) reported a coefficient of variation of 6.1 in the distribution of wealth by using the Survey of Consumer Finances (SCF).[4] The top 5 percents of the population own as much as 53.5% of total wealth, whereas the lowest 40 percents own only 1.35% of it. As explained by Krusell and Smith (1997), it is not possible to represent such an heterogeneity by a two-class society. This is potentially an important problem, because asset prices may end up being determined by a very small group of investors when there is a long tail in the distribution of wealth. We then alternatively approximate the actual distribution of U.S. households by assuming three social classes.[5] There are three social classes: 40% of poors, 5% of riches, and the remaining 55% are in the middle class. Each poor owns $1.35/0.4 = 3,375\%$ of the GDP per capita in each state. Each individual in the middle class gets 82% of the GDP per capita, and each rich agent owns 10.7 time the GDP/cap. We assume again a utility function in the family given by (17). In figure (1), we represent the equity premium as a function of parameter b. When absolute risk tolerance is linear ($b = 1$), the equity premium is 0.24% per year. When $b = 2$, it is virtually

[4]There is some ambiguity about whether we should use the distribution of wealth or income. In an economy with efficient markets, the individual wealth that we should consider is the sum of the financial wealth and the discounted value of future incomes (human capital). The existence of a liquidity constraint makes it probably better to only account for the financial wealth.

[5]For the sake of conciseness, we do not report our simulations for finer wealth classes, which yield similar results.

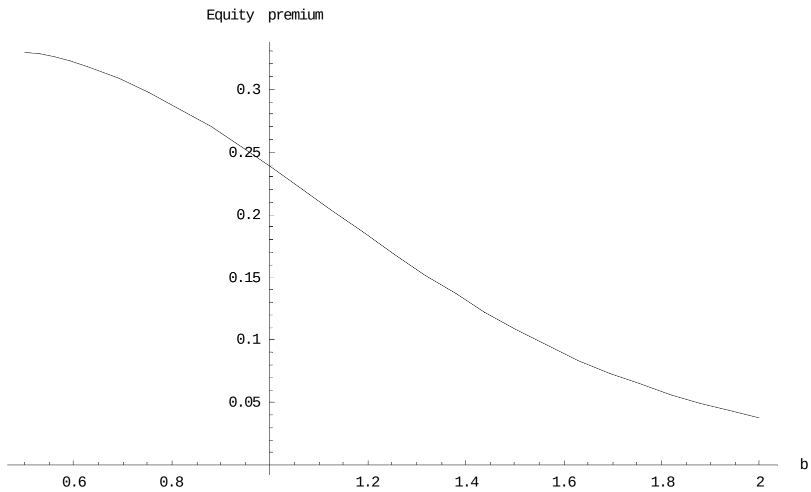

Figure 1: The equity premium (in % per year) with three social classes, and $u'(z) = \exp(-2\frac{z^{1-b}}{1-b})$.

null, because the almost risk-neutral riches almost completely insure the two other classes. When $b = 0.5$, the equity premium culminates to 0.33%, which is 27% larger than the equity premium in the egalitarian economy. This is not a marginal effect, but we are still far to solve the equity premium puzzle.

6 The risk-free rate

We now turn to the analysis of the impact of wealth inequality on the risk-free rate. Weil (1989) showed that the standard asset pricing model does not only perform badly to predict the equity premium, but also to predict the risk free rate. Whereas the calibration of the model provides an estimate of the risk free rate around 4% per year, the observed risk-free rate over the last century has benn around 1%.[6] This difference between the theory and facts raises several question. In particular, should we use the larger theoretical risk-free rate to discount costs and benefits of public investment projects? The answer

[6] See for example Kocherlakota (1996).

to this kind of question depends upon the source of the discrepancy between the theory and the observation. It is interesting in particular to determine whether this discrepancy can be explained by wealth inequality.

The risk-free asset provides one unit of the consumption good at $t = 1$ with probability 1. The price to be paid at $t = 0$ for that asset is

$$\frac{1}{R} = \frac{\sum_{s=1}^{S} \pi_s}{\pi_0} = \frac{\sum_{s=1}^{S} q_s \hat{u}'(z_s)}{q_0 \hat{u}'(z_0)}, \tag{18}$$

where R is the gross risk-free rate (one plus the risk-free rate). It yields

$$R = \frac{\hat{u}'(z_0)}{\beta E \hat{u}'(\tilde{z})}, \tag{19}$$

The level of the risk-free rate depends upon three different characteristics of the representative agent's preference. First, the risk-free rate is affected by the pure preference for the present characterized by $1/\beta$. Assets yielding payoffs in the distant future are less valued to compensate investors for their patience. Second, the risk-free rate depends upon the growth of the representative agent's consumption. If $\hat{u}$ is concave, smoothing consumption over time is valuable. In the absence of uncertainty about the positive growth of consumption, increasing the risk-free rate above $1/\beta$ is necessary in order to compensate investors for not smoothing their consumption over time. By how much the risk-free rate must be increased depends upon the elasticity of intertemporal substitution of the representative agent. Third, if the representative agent is prudent and if future consumption is uncertain, a reduction in the risk-free rate is required in order to convince agents not to accumulate wealth for a precautionary saving motive. The size of this effect depends upon the degree of absolute prudence of the representative agent.

In order to address the question of the effect of wealth inequality on the risk-free rate, we must determine its impact on each of these three motives to save. We do this by starting with a simple case that will be extended in three subsequent steps. This benchmark case is when there is no uncertainty on future GNP per head (to exclude the precautionary saving motive) and no growth in aggregate consumption (to exclude the income smoothing effect). In that case, credit markets will be at work at date 0 to smooth all consumption plans at equilibrium in the unequal economy. This equilibrium is sustained by a gross risk-free rate equaling $1/\beta$. Wealth inequality has no effect on it. We now introduce growth.

6.1 The consumption smoothing effect

In this paragraph, we suppose that $z_1 = z_2 = ... = z_S > z_0$, i.e. there is no uncertainty about the positive growth of the GNP per head. In this economy, insurance markets will be at work to eliminate all individual risks. Because growth is certain and positive, the concavity[7] of $\widehat{u}$ in equation (19) implies that R is larger than $1/\beta$. This is the pure income smoothing effect. Because the level of wealth of agents affects their willingness to smooth consumption over time, we may expect that wealth inequality affects the equilibrium risk-free rate in this economy.

Since the saving decision problem is, technically speaking, a special case of the static portfolio problem with two states of nature, we can use the results developed in sections 2 and 3. First, parallel to Lemma 1, we can show that the intensity of the representative agent's willingness to smooth consumption is measured by the degree of concavity of $\widehat{u}$, i.e., to $-\widehat{u}''(.)/\widehat{u}'(.)$. Second, parallel to Lemma 2, we obtain that wealth inequality raises the intensity of the willingness to smooth consumption if the absolute risk tolerance of u is concave. If the risk-free rate is let unchanged, that would reduce aggregate saving. To compensate for this, the risk-free rate should be increased. This proves the following Corollary of Proposition 1.

Corollary 1 *Suppose that there is no uncertainty about the growth of GNP per head. Suppose also that the growth is positive ($R > 1/\beta$). If absolute risk tolerance is concave (resp. convex), then the equilibrium interest rate in an unequal economy is larger (resp. smaller) than the equilibrium interest rate in the egalitarian one with the same growth. The opposite results hold if growth is negative ($R < 1/\beta$).*

Another way of understanding this result is to observe that the marginal propensity to consume out of wealth is increasing in wealth under certainty

[7]The concavity of $\widehat{u}$ is a direct consequence of conditions (8) and (10).

if and only if absolute risk tolerance is concave. In other words, the concavity of T is equivalent to the concavity of saving with respect to wealth. The Jensen's inequality yields that wealth inequality decreases the average/aggregate saving under this condition. This must be compensated by an increase in the interest rate in order to sustain the equilibrium.

6.2 The precautionary effect

We now turn to the general case with an aggregate uncertainty at date 1. It is useful to decompose the global effect of wealth inequality on the risk-free rate into an consumption smoothing effect and a precautionary effect. We have seen in Corollary 1 that there is no consumption smoothing effect under certainty if R equals $1/\beta$. If we maintain the assumption $R = 1/\beta$ in the uncertain but egalitarian world, we obtain a pure precautionary effect of wealth inequality. This is seen in the following Lemma. It requires the concept of prudence. One is said to be prudent if an increase in risk on future incomes increases savings today. As is well-known, this is true if and only if the third derivative of the utility function on future consumption is positive. Kimball (1990) measures the intensity of precautionary saving by the coefficient of absolute prudence $P(z) = -u'''(z)/u''(z)$. An agent is more prudent than another if the absolute prudence of the first is uniformly larger than the absolute prudence of the other.

Lemma 3 *Suppose that the risk-free rate equals $1/\beta$ in the egalitarian economy. Wealth inequality reduces the risk-free rate if and only if $\hat{u}$ is more prudent than u.*

Proof: We want to prove that $E\hat{u}'(\tilde{z}) \leq \widehat{u}'(z_0)$ whenever $Eu'(\tilde{z}) = u'(z_0)$. Function $\hat{u}$ is more prudent than function u if and only if $\hat{u}'(z) = \psi(u'(z))$ for all z, with ψ increasing and convex. From this, we get that

$$E\hat{u}'(\tilde{z}) = E\psi(u'(\tilde{z})) \geq \psi(Eu'(\tilde{z})) = \psi(u'(z_0)) = \hat{u}'(z_0).$$

This proves sufficiency. Necessity is obtained in a similar way, by contradiction. ■

Thus, to determine whether wealth inequality reduces the risk-free rate, we have to examine how it affects the degree of prudence of the representative agent. This is in parallel with what we have done in the previous section where we had to look at the degree of risk aversion of the representative agent. But contrary to this first result, this new problem is much more complex. Indeed, from condition $\hat{u}'(z) = u'(c_i(z))/\lambda_i$ and after some tedious manipulations, one can verify that

$$\hat{u}''(z) = \lambda_i^{-1} u''(c_i(z))c_i'(z) = -\frac{nu'(c_i(z))}{\lambda_i \sum_j T(c_j(z))},$$

and

$$\hat{u}'''(z) = \frac{n^2 u'(c_i(z)) \sum_j [T(c_j(z))]^2 P(c_j(z))}{\lambda_i [\sum_j T(c_j(z))]^3}.$$

This yields

$$-\frac{\hat{u}'''(z)}{\hat{u}''(z)} = n \frac{\sum_j [T(c_j(z))]^2 P(c_j(z))}{[\sum_j T(c_j(z))]^2}. \tag{20}$$

On the left-hand side of this equation, we have the degree of absolute prudence of the representative agent that we want to be larger than $P(z)$ according to Lemma 3. We learn from this equality that the degree of prudence of the representative agent is a rather complex function of the degree of prudence of the original agent. In particular, contrary to what we got for the degree of risk tolerance, the degree of prudence (or its inverse) of the representative agent is *not* a weighted sum of the degree of prudence (or its inverse) of the original agent measured at different levels of consumption.

To solve this problem, let us define the precautionary equivalent consumption C of consumption plan $(c_1, ..., c_S)$ at date 1 as

$$C(c_1, ..., c_S) = u'^{-1}(\sum_{s=1}^{S} p_s u'(c_s)) \tag{21}$$

The precautionary equivalent consumption is the sure level of consumption that would yield the same marginal utility as the uncertain consumption prospect. It could also be interpreted as today's consumption if the interest rate were equal to the rate of pure preference.

Lemma 4 *Suppose that $R = 1/\beta$ in the egalitarian economy. Then, wealth inequality reduces (resp. increases) the risk-free rate if the precautionary equivalent consumption C $(c_1, ..., c_S)$ defined by (21) is concave (resp. convex).*

Proof: See the Appendix.

The essence of this result comes from the observation that the precautionary equivalent consumption of the representative agent is the mean value of the individual precautionary equivalent consumptions. By Jensen's inequality, it yields that wealth inequality reduces the future precautionary equivalent of the representative agent. It increases his willingness to save. That must be compensated by a reduction of the risk-free rate to sustain the equilibrium.

The following Lemma is useful to link the concavity of C to some properties of the utility function. After writing a first version of this paper, Ben Polak pointed out to us that part 1 of this Lemma is Theorem 106 in Hardy, Littlewood and Polya (1934). Polak (1996) provides another proof of this result and applies it to the problem of the social cost of risk in an unequal economy.[8]

Lemma 5 *Consider a function g from R to R, twice differentiable, increasing and concave. Consider a vector $(q_1, ..., q_m) \in R_+^m$ with $\sum_{j=1}^n q_j = 1$, and a function f from R^m to R, defined as*

$$f(x_1, ..., x_m) = g^{-1}(\sum_{j=1}^{m} q_j g(x_j)).$$

Define function h such that $h(t) = -\frac{g'(t)}{g''(t)}$. We have:

1. *f is concave if h is concave;*

2. *f is neither concave nor convex if h is convex.*

Proof: See the Appendix.

[8] See also Gollier (1996).

Let us now define the absolute degree of imprudence I as the inverse of absolute prudence: $I(z) = P^{-1}(z) = -u''(z)/u'''(z)$. Using this lemma with $g(t) = -u'(t)$ yields that the precautionary equivalent consumption is concave in the vector describing the consumption plan if the inverse of absolute prudence is concave. Combining this with Lemma 4, we directly obtain the following Proposition:

Proposition 3 *Suppose that the original agent is prudent ($u''' \geq 0$) and that $R = 1/\beta$ in the egalitarian economy. Then, wealth inequality reduces the equilibrium risk-free rate, if absolute imprudence ($I(z) = -u''(z)/u'''(z)$) is concave.*

■

Proposition 3 states that, if the inverse of absolute prudence is positive and concave, then wealth inequality has a negative precautionary effect on the risk-free rate. Thus, the concavity of the inverse of absolute prudence may explain why the representative agent à la Lucas overpredicts the rate of return on safe bonds if wealth inequality is not taken into account. Up to our knowledge, this is the first example of application of the concept of imprudence. Whether it is concave or convex with wealth is an open question.

We are now in a situation to combine the consumption smoothing effect with the precautionary effect to obtain a general picture of the effect of wealth inequality on the equilibrium risk-free rate. Our findings are summarized in the following Proposition.

Proposition 4 *Suppose that the risk-free rate is less (resp. larger) than $1/\beta$ in the egalitarian economy. Then, wealth inequality reduces the risk-free rate if the two following conditions are satisfied:*

1. *the absolute imprudence is concave;*
2. *the absolute tolerance to risk is concave (resp. convex).*

Proof: See the Appendix.

Whether the equilibrium risk-free rate in the egalitarian economy is larger or smaller than the rate of pure preference for the present depends upon whether the willingness to smooth consumption in a growing economy overweights the willingness to accumulate precautionary saving because of the uncertainty affecting this growth. This is mainly an empirical question.

If we consider an egalitarian economy whose risk-free rate is small, we conclude that under the concavity of both the inverse of the absolute risk aversion and the inverse of absolute prudence, the equity premium will be underpredicted and the risk-free rate will be overpredicted by a calibrator who does not take into account of wealth inequality. However, these two results are not entirely parallel, since the first condition is necessary and sufficient for an underprediction of the equity premium, whereas the second condition is only sufficient for the overprediction of the risk-free rate. It is in no way necessary, since it is always possible to find a function u with a positive and *convex* inverse of absolute prudence, a distribution of wealth and a discount factor such that the unequal distribution of wealth also reduces the risk-free rate.

7 Asset pricing with background risk and HARA preferences

All existing models trying to solve the puzzles of asset prices uses CRRA preferences for which we know that wealth inequality does not affect equilibrium prices. Still, in several recent papers, as in Constandinides and Duffie (1996), Heaton and Lucas (1996) and Krusell and Smith (1997), it is claimed that taking into account of wealth inequality has an important effect on prices. In this section, we try to reconcile this two a priori opposite conclusions by introducing incomplete markets, an important ingredient of the above-mentioned models.

That markets are incomplete is an obvious fact of life. All tests using disaggregated consumption data clearly reject the hypothesis that markets are complete (see Attanasio and Davies (1993) and Townsend (1994) for example). Weil (1992) has shown that the introduction of an uninsurable labour income risk to the representative CRRA agent model can help to solve the puzzle. The question that we raise in this section is whether adding wealth inequality to this construction does provide another ingredient that goes into the good direction. We build our answer to this question in three steps. First, we prove that wealth inequality raises the equity premium and reduces the risk free rate when the uninsurable background risk is small, for all HARA utility functions. Second, we show that this result is reversed for larger background risks. Third, and most importantly, we show that the effect of wealth inequality on prices is at best marginal. This raises doubt about the relevance of the interaction between wealth inequality and incomplete markets to explain puzzles in asset prices.

Suppose that consumer i is endowed in every state s with a marketable wealth ω_{is} plus an uninsurable, non-marketable risk $\widetilde{y}$ that is independent of the state. This risk is idiosyncratic in the population of consumers. As is standard in the background risk literature, the presence of this independent risk is taken into account in the model by defining an indirect utility function v that is defined as

$$v(z) = Eu(z + \widetilde{y})$$

for all z. The economy with agents having a utility function u and facing the idiosyncratic risk $\widetilde{y}$ is equivalent to an economy with agents having a utility function v and no background risk. We know from the previous sections that the effect of wealth inequality on the equity premium and the risk-free rate depends upon the concavity of the inverses of absolute risk aversion and absolute prudence. The absolute risk tolerance of v equals

$$T_v(z) = -\frac{Eu'(z + \widetilde{y})}{Eu''(z + \widetilde{y})}$$

when evaluated at z. Differentiating with respect to z yields

$$T_v'(z) = \frac{Eu'(z + \widetilde{y})Eu'''(z + \widetilde{y})}{\left[Eu''(z + \widetilde{y})\right]^2} - 1.$$

It implies that T_v is concave if

$$\begin{aligned} K(z) \quad = \quad & \left[Eu''(z + \widetilde{y})\right]^2 Eu'''(z + \widetilde{y}) + Eu'(z + \widetilde{y})Eu''(z + \widetilde{y})Eu''''(z + \widetilde{y}) \\ & -2Eu'(z + \widetilde{y})\left[Eu'''(z + \widetilde{y})\right]^2 \end{aligned}$$

is nonnegative. Similarly, the inverse of absolute prudence is concave if

$$\begin{aligned} H(z) &= \left[Eu'''(z+\widetilde{y})\right]^2 Eu''''(z+\widetilde{y}) + Eu''(z+\widetilde{y})Eu'''(z+\widetilde{y})Eu'''''(z+\widetilde{y}) \\ &\quad -2Eu''(z+\widetilde{y})\left[Eu''''(z+\widetilde{y})\right]^2 \end{aligned}$$

We are not aware of any sufficient condition for these properties to hold other than the ones presented in the next Proposition.

Proposition 5 *Suppose that the utility function u is HARA and that the background risk $\tilde{y}$ is small relative to wealth z. Then, $K(z)$ and $H(z)$ are nonnegative. It implies that ex ante wealth inequality raises the equity premium. If R is less than $1/\beta$ in the egalitarian economy, ex ante wealth inequality also reduces the risk-free rate.*

Proof: Without loss of generality, we suppose that the expectation of $\widetilde{y}$ is zero. Let $\widetilde{y}$ be distributed as $k\widetilde{x}$, with $E\widetilde{x} = 0$ and $Var(\widetilde{x}) = 2$. Then, a second order Taylor expansion around z yields

$$Eu^{(n)}(z + k\widetilde{x}) = u^{(n)}(z) + k^2u^{(n+2)}(z) + o(k^3).$$

It implies that

$$\begin{aligned} K(z) &= (u^{(2)}(z) + k^2u^{(4)})^2(u^{(3)}(z) + k^2u^{(5)}) \\ &\quad +(u^{(1)}(z) + k^2u^{(3)})(u^{(2)}(z) + k^2u^{(4)})(u^{(4)}(z) + k^2u^{(6)}) \\ &\quad -2(u^{(1)}(z) + k^2u^{(3)})(u^{(3)}(z) + k^2u^{(5)}) + o(k^3), \end{aligned}$$

or, equivalently,

$$K(z) = a_0 + a_1k^2 + o(k^3).$$

It is easy to verify that $a_0 = 0$ (this is due to the linearity of T) and

$$a_1 = (u^{(2)})^2u^{(5)} + 3u^{(2)}u^{(3)}u^{(4)} + u^{(1)}(u^{(4)})^2 + u^{(1)}u^{(2)}u^{(6)} - 2(u^{(3)})^3 - 4u^{(1)}u^{(3)}u^{(5)}$$

where $u^{(n)} = u^{(n)}(z)$. Following the assumption that preferences are HARA, suppose that u is defined by condition (11). Let x denote $a + \frac{z}{\gamma}$. It implies that

$$u^{(1)} = x^{-\gamma};\ u^{(2)} = -x^{-\gamma-1};\ u^{(3)} = \frac{\gamma+1}{\gamma}x^{-\gamma-2};\ u^{(4)} = \frac{(\gamma+1)(\gamma+2)}{\gamma^2}x^{-\gamma-3};...$$

After some tedious manipulations, we obtain that

$$a_1 = \frac{4(\gamma+1)x^{-3\gamma-6}}{\gamma^4} > 0.$$

We conclude that $K(z)$ is positive is k is small enough. The same computations can be made for H by replacing γ by $\gamma - 1$. ■

What is intuition for background risk to concavify absolute risk tolerance in the HARA case? To get this intuition, notice first that a zero-mean background risk has the effect to reduce the degree of risk tolerance: $T_v(z) \leq T_u(z)$ for all z. This is because HARA functions are risk vulnerable, a concept introduced by Gollier and Pratt (1996) to guarantee that any zero-mean background risk raises the aversion towards any other independent risk. This corresponds to the natural hypothesis that independent risks are substitutes rather than complement. Now, it seems natural to assume that the adverse effect of a given background risk on the degree of tolerance towards the independent portfolio risk is smaller for poor agents than for wealthy agents. This is true only if $T'_v(z)$ is less than $T'_u(z)$, a condition that is difficult to check by using the standard methods of the economics of uncertainty. We show however in the next Proposition (proven in the Appendix) that this condition holds when u is HARA and when the background risk is small.

Proposition 6 *Suppose that u is HARA and that the background risk is small. It implies that the absolute risk tolerance of wealthier people is less adversely affected by background risk: $T'_v(z) > T'_u(z)$.*

Under HARA, we conclude that T_v is uniformly less than the linear function T_u, and tends monotonically to this function for large wealth levels relative to the size of background risk. This tells us that the risk tolerance of v must be concave for large relative wealth levels, i.e., for small background risks. The concavity of T_v implies that wealth inequality raises the equity premium. Notice however that this result does not tell us anything about the concavity or convexity of T_v when the background risk is large with respect to wealth z.

Proposition 5 is related to the result by Constantinides and Duffie (1996), who showed that a large equity premium can be generated by wealth inequality in an economy with "*judiciously*" modeled uninsurable income uncertainty. In order to quantify the effect of wealth inequality in an economy with uninsurable incomes and CRRA preferences, we conducted a simulation based on the same macroeconomic uncertainty as in section 5. As a reminder, we have two possible states, with $z_1 = 0.99$, $z_2 = 1.0632$ and $p_1 = 0.61756$. Now, each agent bears a lottery ticket with a fifty-fifty chance to win or lose $k = 0.25$. This simulates a large risk on labour incomes. In the egalitarian economy, the equity premium is equal to 0.281%. When there are two classes of equal size where the poors and the riches are endowed with respectively 50% and 150% of the GDP per capita, the equity premium goes up to 0.285%. In accordance with the above Proposition, wealth inequality raises the equity premium, but it accounts for around one-thousandth of a percent in the equity premium! This leads us to conclude that there is no hope that wealth inequality really matters in models with incomplete markets and CRRA preferences. Krusell and Smith (1997) obtained the same conclusion by using a much richer and sophisticated simulation model, including infinite horizon, borrowing constraints, and persistence in shocks on labour incomes. To sum up, our Proposition 5 together with Constandinides and Duffie (1996) provides a ground for relying on wealth inequality and income uncertainty to rationalize observed equity premia. However, our simulations together with those of Krusell and Smith (1997) indicate that the effect of wealth inequality is at best a second order effect, when absolute risk tolerance is linear.

When the background risk is small, the effect of wealth inequality on asset prices is small. One could believe that we just need to increase the degree of income uncertainty to get a more sizeable effect similar to the one suggested by Constantinides and Duffie (1996). This intuition is wrong, however. Indeed, our analysis provides a more surprising result, apparently unnoticed before, that wealth inequality may even *reduce* the equity premium in an economy with HARA preferences and a large income uncertainty! This is shown by the following example: take a utility function with a constant relative risk aversion of 2 and let us consider background risk $(-k, 1/2; k, 1/2)$. We have drawn in Figure 2 the equity premium in the

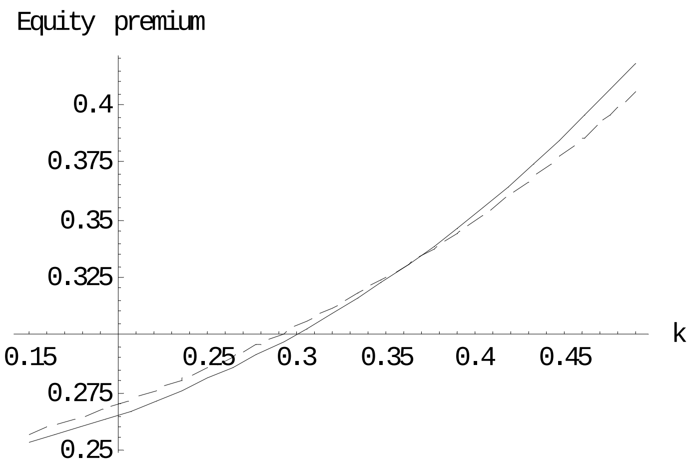

Figure 2: The equity premium as a function of the size of the income risk. The dotted (plain) line is for the unequal (egalitarian) economy.

economy with $z_1 = 0.99$, $z_2 = 1.0632$, $p_1 = 0.61756$, and a distribution of wealth with two social classes of equal size. Each poor is endowed with 50% of the GDP/cap, whereas the riches are endowed with 150% of the GDP/cap. The effect of wealth inequality on the equity premium is positive for low income uncertainty, with a maximum around $k = 0.25$, but wealth inequality *reduces* the equity premium for larger income uncertainties. We conclude that the Constantinides and Duffie's result cannot come from the assumption of a large background risk.[9] It could be explained by their assumption that the income risk is correlated with the portfolio risk, or with the social status of investors.[10]

The main restriction of the model that we discussed in this section is its two-period structure. As long as we maintained the assumption that markets are complete, limiting the model to two periods was without loss of generality. But once the uninsurability of the risk associated to the flow of labour incomes is recognized, the two-period modeling becomes restric-

[9]The assumed persistence of income shock is one way to model large, undiversifiable, income risks.

[10]If poorer investors have a larger income risk, that will raise the concavity of the T_v function, yielding a larger equity premium.

tive. However, we have not be able to obtain theoretical results beyond two periods. Technically, adding a third period raises the question of whether the Bellman function inherits the property of the concavity of the absolute risk tolerance/imprudence of the original utility function. There is no hope to solve this problem given the current level of our knowledge in the economics of uncertainty, except in very special cases as the one examined by Constantinides and Duffie (1996).

8 Conclusion

In order to evaluate the equity premium and the risk-free rate, economists use to calibrate a model à la Lucas with a family of utility functions exhibiting linear risk tolerance (HARA) which entails exponential, power and logarithmic functions. When consumers are HARA and face no background, uninsurable, risk, then the distribution of wealth

- has no effect on the equity premium;
- has no effect on the risk-free rate.

In this paper, we inquired about whether not taking into account of the unequal distribution of wealth may explain the equity premium puzzle and the risk-free rate puzzle. Obviously, it cannot explain these paradoxes if we believe in HARA functions and complete insurance markets. We show in this paper that an unequal distribution of wealth in a complete markets economy

- increases the equity premium if and only if the absolute risk tolerance is concave; and
- reduces the risk-free rate if the absolute imprudence is concave.

Whether these conditions are satisfied or not remains an empirical question. We presented some stylized facts that raise some doubt in particular about the hypothesis of a concave absolute risk tolerance. But even if researchers are convinced to depart from HARA, they will not resolve the puzzles of asset prices by introducing wealth inequality alone, as shown by our simulations.

Finally, we examined the effect of wealth inequality in a model that has been frequently used in recent years, i.e. a model with a linear absolute risk tolerance but including an idiosyncratic background risk. We proved that the presence of a small uninsurable background risk biases asset prices towards a larger equity premium and a smaller risk free rate. However, the attitude towards time and risk of Society is only marginally affected by wealth inequality in this paradigm with HARA utility functions and an idiosyncratic background risk. This conclusion is similar to those obtained in more sophisticated calibrations, as in Krusell and Smith (1997) for example. The problem of determining whether wealth inequality is important or not to explain puzzles in asset pricing is now transferred to the question of how do risk tolerance and imprudence change with wealth.

References

Attanasio, O., and S.J. Davies, (1993), Relative Wage Movements and the Distribution of Consumption, Working Paper, Stanford.

Bertaut , C.C., and M. Haliassos, (1997), Precautionary portfolio behavior from a life-cycle perspective, *Journal of Economic Dynamics and Control*, 21, 1511-1542.

Carroll, C.D., (1998), Why Do the Rich Save so Much?, discussion paper, The Johns Hopkins University.

Carroll, C.D. and M.S. Kimball, (1996), On the concavity of the consumption function, *Econometrica*, 64, 981-992.

Cocco, J.F., F.J. Gomes and P.J. Maenhout, (1998), Consumption and portfolio choice over the life-cycle, Discussion paper, Harvard University.

Constantinides, G. M. (1982), Intertemporal asset pricing with heterogenous consumers, and without demand aggregation, *Journal of Business*, 55, 253-267.

Constantinides, G.M., and D. Duffie, (1996), Asset Pricing with Heterogeneous Consumers, *Journal of Political Economy*, 104, 219-240.

Diaz-Gimenez, J, V. Quadrini and J.-V. Rios-Rull, (1996), Measuring inequality: Facts on the distributions of earnings, income, and wealth, Discussion Paper, University of Pennsylvania.

Gollier, C. and J.W. Pratt, (1996), Risk vulnerability and the tempering effect of background risk, *Econometrica*, 64, 1109-1124.

Gollier, C. and R.J. Zeckhauser, (1997), Horizon Length and Portfolio Risk, Discussion Paper, University of Toulouse.

Gollier, C., (1996), Wealth Inequality: Its Impact on the Equity Premium, Growth, and Dynamic Investment Strategies, mimeo, University of Toulouse.

Gollier, C. and J.W. Pratt, (1996), Risk vulnerability and the tempering effect of background risk, *Econometrica*, 64, 1109-1124.

Guiso, L., T. Jappelli and D. Terlizzese, (1996), Income Risk, Borrowing Constraints, and Portfolio Choice, *American Economic Review*, 86, 158-172.

Guiso, L., and M. Paiella, (2000), Risk aversion, wealth and financial market imperfections, mimeo, University College London.

Hansen, L.P. and K.J. Singleton, (1982), Generalized instrumental variables estimation of nonlinear rational expectations models, *Econometrica*, 50, 1269-1286.

Hardy, Littlewood and Polya, (1934), *Inequalities*, reprinted in 1997 by Cambridge University Press.

Heaton, J., and D.J. Lucas, (1995), The Importance of Investor Heterogeneity and Financial Market Imperfections for the Behavior of Asset Prices, *Carnegie-Rochester Conference Series on Public Policy*, 42, 1-33.

Heaton, J. and D.J. Lucas, (1996), Evaluating the effects of incomplete markets on risk sharing and asset pricing, *Journal of Political Economy*, 104, 443- 487.

Hennessy, D.A., and H.E. Lapan, (1998), On the nature of certainty equivalent functionals, discussion paper, Iowa State University.

Jagannathan, R., and N.R. Kocherlakota, (1996), Why should older people invest less in stocks than younger people, Federal Reserve Bank of Minneapolis Quarterly Review, 20, 11-23.

Kandel, S. and R.F. Stambaugh, (1991), Asset returns and intertemporal preferences, *Journal of Monetary Economics*, 27, 39-71.

Kimball, M.S., (1990), Precautionary Saving in the Small and in the Large, *Econometrica*, 58, 53-73.

Kimball, M.S., (1993), Standard Risk Aversion, *Econometrica*, 61, 589-611.

Kimball, M.S., and C.D. Carroll, (1996), On the Concavity of the Consumption Function, Econometrica, 64, 981-992.

Kocherlakota, N.R., (1996), The Equity Premium: It's Still a Puzzle, *Journal of Economic Literature*, 34, 42-71.

Krusell, P., and A.A. Smith, (1997), Income and Wealth Heterogeneity, Portfolio Choice, and Equilibrium Asset Returns, *Macroeconomic Dynamics*, 1, 387-422.

Leland, H.E., (1980), Who Should Buy Portfolio Insurance? *Journal of Finance*, 35, 581-596.

Lucas, (1978), Asset Prices in an Exchange Economy, *Econometrica*, 46, 1429-1446.

Lusardi, A., (1992), Permanent income, current income, and consumption: Evidence from panel data, Discussion paper, Darmouth College.

Maddison, A., (1991), *Phases of Economic Development*, Oxford Economic Press.

Mehra, R. and E. Prescott, (1985), The Equity Premium: A Puzzle, *Journal of Monetary Economics*, 10, 335-339.

Mossin, J., (1968), Optimal multiperiod portfolio policies, *Journal of Business*, 215-229.

Polak, B., Notes on Anand and Sen's Notes on Equally Distributed Equivalent Incomes, mimeo, Yale University.

Poterba, J., and A. Samwick, (1997), Household portfolio allocation over the life cyce, NBER WP, n. 6185.

Raviv, A. (1979). The Design of an Optimal Insurance Policy, *American Economic Review* 69, 84-96.

Townsend, R.M., (1994), Risk and Insurance in Village India, *Econometrica*, 62, 539-591.

Vissing-Jorgensen, A., (1999), Towards an explanation of household portfolio choice heterogeneity: Nonfinancial income and participation cost structure, Discussion paper, University of Chicago.

Weil, P., (1989), The equity premium puzzle and the risk free rate puzzle, *Journal of Monetary Economics*, 24, 401-21.

Weil, P., (1992), Equilibrium Asset Prices with Undiversifiable Labor Income Risk, *Journal of Economic Dynamics and Control*, 16, 769-789.

Wilson, R., (1968), The Theory of Syndicates, *Econometrica*, 36, 113-132.

Appendix: Proof of Lemma 1

In an egalitarian economy, Π satisfies the following condition:

$$E(\tilde{z}-\Pi)u'(\tilde{z})=0,$$

where E is the expectation operator and $\tilde{z}$ is the random variable of average consumption at $t=1$. Thus, wealth inequality reduces the price of equity if

$$E(\tilde{z}-\Pi)\hat{u}'(\tilde{z})\leq 0.$$

Suppose that $\hat{u}$ is more concave than u, i.e., there exists an increasing and concave function ϕ such that $\hat{u}(z)=\phi(u(z))$ for all z. We have the following sequence of (in)equalities:

$$\begin{aligned} E(\tilde{z}-\Pi)\hat{u}'(\tilde{z}) &= E(\tilde{z}-\Pi)\hat{u}'(\tilde{z})\phi'(u(\tilde{z})) \\ &\leq \phi'(u(\Pi))E(\tilde{z}-\Pi)u'(\tilde{z}) \\ &= 0. \end{aligned}$$

The inequality comes from the concavity of ϕ which implies that $(z-\Pi)\phi'(\hat{u}(z))$ is less than $(z-\Pi)\phi'(\hat{u}(\Pi))$ for all z. This concludes the proof of sufficiency. Necessity is obtained by contradiction by selecting $\tilde{z}$ with support in an interval where ϕ is locally convex. ■

Appendix: Proof of Lemma 4

By assumption, we know that the consumption level at date 0 is just equal to the precautionary equivalent consumption at date 1 in this economy:

$$z_0=C(z_1,...,z_S).$$

The risk-free rate in the unequal economy will be less than $1/\beta$ if wealth inequality reduces the precautionary equivalent consumption, i.e., if

$$\hat{u}'^{-1}(\sum_{s=1}^{S} p_s \widehat{u}'(z_s)) \leq C(z_1, ..., z_S)$$

or, equivalently, using conditions $\hat{u}'(z) = u'(c_i(z))/\lambda_i$ and $\sum_i \frac{1}{n} c_i(z) = z$, if

$$\sum_{i=1}^{n} \frac{1}{n} u'^{-1}(\sum_{s=1}^{S} p_s u'(c_i(z_s))) \leq C(z_1, ..., z_S). \tag{22}$$

This is equivalent to

$$\sum_{i=1}^{n} \frac{1}{n} C(c_i(z_1), ..., c_i(z_S)) \leq C(\sum_{i=1}^{n} \frac{1}{n} c_i(z_1), ..., \sum_{i=1}^{n} \frac{1}{n} c_i(z_S)), \tag{23}$$

which means that the precautionary equivalent consumption $C(c_1, ..., c_S)$ is a concave function. ■

Appendix: Proof of Lemma 5

Let us first observe that it is enough to look at the case $m = 2$. Indeed, defining $f_m(\mathbf{x}; \mathbf{q}) = g^{-1}(\sum_{j=1}^{m} q_j g(x_j))$, we have:

$$\begin{aligned} f_3(x_1, x_2, x_3; q_1, q_2, q_3) &= g^{-1}(q_1 g(x_1) + (q_2 + q_3) g(f_2(x_2, x_3; \tfrac{q_2}{q_2+q_3}, \tfrac{q_3}{q_2+q_3}))) \\ &= f_2(x_1, f_2(x_2, x_3; \tfrac{q_2}{q_2+q_3}, \tfrac{q_3}{q_2+q_3}); q_1, q_2 + q_3). \end{aligned}$$

Since $\partial f_m / \partial x_j \geq 0$, applying the result by recurrence would yield the result.

Observe also that

$$\frac{\partial f}{\partial x_j} = \frac{q_j g'(x_j)}{g'(f(\mathbf{x}))}$$

and

$$\frac{\partial^2 f}{\partial x_j^2} = q_j \frac{g''(x_j) g'(f(\mathbf{x})) - g'(x_j) g''(f(\mathbf{x})) \frac{\partial f}{\partial x_j}}{[g'(f(\mathbf{x}))]^2},$$

or, equivalently,

$$\frac{\partial^2 f}{\partial x_j^2} = \frac{q_j}{[g'(f(\mathbf{x}))]^3}\Big[g''(x_j)[g'(f(\mathbf{x}))]^2 - q_j g''(f(\mathbf{x}))[g'(x_j)]^2\Big].$$

Let us define function k as

$$k(g(t)) = -\frac{[g'(t)]^2}{g''(t)}.$$

We can then rewrite the above equality as follows:

$$\frac{\partial^2 f}{\partial x_j^2} = \frac{q_j g''(x_j)g''(f(\mathbf{x}))}{[g'(f(\mathbf{x}))]^3}\Big[q_j k(g(x_j)) - k(g(f(\mathbf{x})))\Big].$$

Similarly, we have

$$\frac{\partial^2 f}{\partial x_i x_j} = -q_i q_j \frac{g'(x_i)g'(x_j)g''(f(\mathbf{x}))}{[g'(f(\mathbf{x}))]^3}.$$

For $m = 2$, the determinant of the Hessian is

$$\left|\frac{\partial^2 f}{\partial \mathbf{x}^2}\right| = q_1 q_2 k(g(f(\mathbf{x})))\frac{g''(x_1)g''(x_2)[g''(f(\mathbf{x}))]^2}{[g'(f(\mathbf{x}))]^6}\Big[k(g(f(\mathbf{x}))) - q_1 k(g(x_1)) - q_2 k(g(x_2))\Big].$$

Since $g(f(\mathbf{x})) = q_1 g(x_1) + q_2 g(x_2)$, with $q_1 + q_2 = 1$, the right-hand side of the above equality is positive (resp. negative) if k is concave (resp. convex). Notice also that

$$k(g(f(\mathbf{x}))) \geq q_1 k(g(x_1)) + q_2 k(g(x_2)) \geq q_j k(g(x_j))$$

if k is positive and concave.

Thus, if k is concave, then

$$\left|\frac{\partial^2 f}{\partial \mathbf{x}^2}\right| \geq 0 \text{and } \frac{\partial^2 f}{\partial x_j^2} \leq 0,$$

i.e. $\frac{\partial^2 f}{\partial \mathbf{x}^2}$ is semi-definite negative, and f is concave. If k is convex, the determinant of the Hessian matrix is negative, implying that f is neither concave nor convex.

Observe now that $k(g(t)) = g'(t)h(t)$ with $h(t) = -g'(t)/g''(t)$. It yields

$$k'(g(t) = h'(t) - 1,$$

and

$$k''(g(t)) = \frac{h''()}{g'(t)}.$$

We conclude that k is concave if and only if h is concave. This concludes the proof. ■

Proof of Proposition 4

Define

$$H(r) = \frac{\hat{u}'(d(r))}{E\hat{u}'(\tilde{z})},$$

where $d(r)$ is defined by

$$\frac{u'(d(r))}{Eu'(\tilde{z})} = r.$$

$d(r)$ is the consumption at $t = 0$ which generates an equilibrium gross risk-free rate equaling r/β. $H(r)/\beta$ is the equilibrium risk-free rate in the corresponding unequal economy. Suppose that there exists a level r_0 such that $H(r_0) = r_0$. Denoting $d_0 = d(r_0)$, one can check that

$$\begin{aligned} H'(r_0) &= \frac{\hat{u}''(d_0)}{E\hat{u}'(\tilde{z})} d'(r_0) \\ &= \frac{\hat{u}''(d_0)}{E\hat{u}'(\tilde{z})} \frac{Eu'(\tilde{z})}{u''(d_0)} \\ &= \frac{r_0\hat{u}''(d_0)}{\hat{u}'(d_0)} \frac{u'(d_0)}{r_0 u''(d_0)} \\ &= \frac{T(d_0)}{\hat{T}(d_0)}. \end{aligned}$$

Suppose that absolute risk tolerance T is concave. Then, from Lemma 2, we know that $\hat{T}(d_0) \leq T(d_0)$, implying that $H'(r_0)$ is larger than 1. In short, under condition $T'' \leq 0$, if curve H crosses the 45-degree line, it must do it from below. But we know that $H(1) \leq 1$ by Proposition 3. Thus, $H(r)$ can exceed r only for value of r larger than 1. Consequently, under the concavity of absolute risk tolerance and absolute prudence, wealth inequality may not increases the risk-free rate when the initial rate is less than $1/\beta$. A symmetric argument can be used for the case of a convex absolute tolerance to risk. ■

Appendix: Proof of Lemma 6

We limit the proof to the CRRA case. We have to prove that $T'_v(z)$ is larger than $T'_u(z)$, or equivalently, that

$$\frac{P_v(z) - P_u(z)}{P_u(z)} \geq \frac{A_v(z) - A_u(z)}{A_u(z)}, \tag{24}$$

where A_h and P_h are the degree of absolute risk aversion of respectively function h and $-h'$. Because $\widetilde{y}$ is assumed to be small, we may use the approximation given in Gollier and Pratt (1996, equation (11)) to obtain that

$$\frac{A_v(z) - A_u(z)}{A_u(z)} = \frac{\sigma^2_{\widetilde{y}}}{z^2}(\gamma + 1), \tag{25}$$

where γ is the degree of relative risk aversion of u. Notice that $-u'$ is also CRRA with a constant relative risk aversion $\gamma + 1$. We can thus perform exactly the same approximation for P_v, replacing γ by $\gamma + 1$, to obtain

$$\frac{P_v(z) - P_u(z)}{P_u(z)} = \frac{\sigma^2_{\widetilde{y}}}{z^2}(\gamma + 2). \tag{26}$$

Cobining the above three conditions yields the result.■