The Libertarian Identification Rule in Finite Atomistic Lattices

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Abstract

Collective Identification Procedures (CIPs) model admission rules regulating membership in associations, communities and clubs: the *Libertarian identification rule* F^l is the CIP which essentially relies on self-certification. This paper studies F^l in an arbitrary finite atomistic lattice, allowing an unified treatment of collective identification problems with either *exogenous* or *endogenous* choice of classification labels. An axiomatic characterization of F^l in that general setting is provided and contrasted with previously known characterizations which only work in more specialized (e.g. distributive) lattices, and are therefore confined to collective identification problems with exogenously fixed labels. Non-manipulability properties of F^l on a certain simple restricted domain are also considered and shown to hold in any finite atomistic lattice.

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1 Introduction

Collective identification procedures (CIPs) model the admission rules that are used in order to identify the legitimate members of certain formal or informal associations, clubs, or communities, treating such rules as opinion-aggregating functions. Given a certain population, each agent submits an assessment of membership qualifications concerning the entire population, and a CIP amalgamates the resulting profile of assessments to establish who is to be considered a member. In the last decade, following the lead of a seminal paper by Kasher and Rubinstein (see Kasher and Rubinstein (1997)) some work has been devoted to the formal social-choice-theoretic study of CIPs. The extant literature has been mostly focussed on classifications with exogenously given labels (and in fact, on *binary* labels, with 'being a J(ew)' -the original issue addressed by Kasher and Rubinstein- mostly acting as a paradigmatic case: this version of the identification problem will be denoted here as the *standard binary classification problem*). Here, we are interested in the the self-certification-based 'libertarian' rule. In general terms, this rule may described thus: for any possible opinion profile, *the resulting associative structure is the one engendered by those agents who self-certify their qualifications to join it as members,* namely *the smallest associative structure that includes all willing, self-certifying agents.* The libertarian rule and its characterizations have attracted much attention, playing a central role in the literature as a prominent benchmark (see e.g. Samet and Schmeidler (2003), Sung and Dimitrov (2005), Miller (2006), Houy (2007), and of course Kasher and Rubinstein (1997)). Indeed, whenever population units are to be classified according to a *prefixed* set of exhaustive labels, either binary such as member/nonmember or not, the libertarian rule simply states that each agent is classified under a certain label if *and only if* that label is attached by that agent to itself. Put otherwise, under the libertarian rule a) providing self-certification ensures membership (positive effectiveness of self-certification), while b) denying self-certification prevents it (negative effectiveness of self-certification). However, one might also want to consider the case of a *fully endogenous self-classification problem*, where agents simultaneously decide *memberships and the set of relevant (mutually incompatible) classes,* with corresponding labels[1]. Apparently, such a case can be modelled

[1]In a more formal vein, *exogenous labelling* denotes use of a set of labels which is unrelated to population parameters, while on the contrary *endogenous labelling* entails use of a set of admissible labels whose cardinality does depend on population size.

as follows: each population unit (or rather each pair of population units) proposes a partition of the population, and a CIP aggregates the resulting profile of partitions to produce a final partition: here, the relevant labels are the blocks of the latter, i.e. their extensions. Unfortunately, it turns out that, as it is easily checked, no CIP for partitions can ensure both positive and negative effectiveness of self-certification: if two pairs A, B of population units think they should stay together and pair C also wish to join them within the same block, but pair A agree while B refuse, then there is no way to accommodate all the relevant self-certificatory claims in the final partition.

But then, *does there exist any appropriate formulation of the libertarian identification rule which works for the standard binary* and *the fully endogenous classification problems?*

This paper addresses this issue pursuing the analysis within the *general framework of an arbitrary atomistic lattice.* In fact, an atomistic lattice is by definition a lattice with a minimum whose elements are all composed (i.e. joins) of atoms (an atom is an element 'just' greater than the minimum i.e. greater than the minimum but with no elements in between). When -as in the collective identification setting- lattices model coalition structures, atomistic lattices correspond to those coalition structures which can be reduced to some basic nonnegligible agents.

Now, the natural settings for the standard binary classification problem (for a finite population) and for the fully endogenous classification are, respectively, the (boolean) lattice of subsets of the (finite) set of population units, and the lattice of partitions of the set of population units, *which are both atomistic lattices.*

The libertarian CIP F^l in (finite) atomistic lattices is thus defined, and studied. A characterization of $F^l$is provided and contrasted with previously known characterizations of the libertarian CIP in the more specialized setting of (finite, boolean) distributive lattices. Some basic manipulability and cooperative stability properties of F^l in simple environments with self-oriented preferences are also discussed.

The present paper is organized as follows: Section 2 is devoted to a presentation of the model, and the results. Section 3 includes a discussion of some related literature. Section 4 provides some short comments on the results of Section 2, and remarks about possible extensions of the analysis.

2 Notation, model, and results

Let $\mathbf{L} = (L, \leqslant)$ be a finite *lattice* namely a finite partially ordered set[2] such that for any $x, y \in L$ both the greatest lower bound $x \wedge y$ and the least upper bound $x \vee y$ of $\{x, y\}$ do exist. For any $A \subseteq L$, $\wedge A$ and $\vee A$ are defined in the obvious way by induction on cardinality of A. Clearly, $\mathbf{L}$ has also a minimum $0_L = \wedge L$, and a maximum $1_L = \vee L$. A *join irreducible* element of $\mathbf{L}$ is any $j \in L$ such that $j \neq \wedge L$ and for any $x, y \in L$ if $j = x \vee y$ then $j \in \{x, y\}$. The set of all join irreducible elements of $\mathbf{L}$ is denoted J_L: it is also assumed that $\# J_L \geq 2$ in order to avoid tedious qualifications or trivialities. An *atom* of $\mathbf{L}$ is any $j \in L$ which is an upper cover of 0_L- written $0_L \lessdot j$- i.e. $0_L < j$ and $l = j$ for any $l \in L$ such that $0_L < l \leqslant j$. The set of all atoms of $\mathbf{L}$ is denoted A_L. It is easily checked that in general $A_L \subseteq J_L$, while the converse may not hold[3]. $\mathbf{L}$ is *atomistic* if and only $A_L = J_L$ i.e. equivalently whenever each element $l \in L$ is the least upper bound of a set of atoms. An atomistic lattice $\mathbf{L}$ may or may not be *distributive* i.e. such that $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for any $a, b, c \in L$. This paper will be dealing with both distributive and nondistributive atomistic lattices.

The suggested interpretation in the collettive identification problem is the following: L denote the set of all possible associative structures, and A_L the set of all their basic constitutive units, i.e. the set of relevant agents in the collective identification process under consideration. For any associative structure $a \in L$, and atom $i \in A_L$, $i \leqslant a$ denotes that i is a *component* of a.

A *Collective Identification Procedure (CIP)* on A_L is a function $F : L^{A_L} \rightarrow L$. In particular, for any $i, j \in A_L$, it will be said that j *accepts/nominates* i at opinion profile $x = (x_1, ..., x_{\# A_L})$ whenever $i \leqslant x_j$. At each opinion profile x, $F(x)$ denotes the resulting associative structure, whose set of basic components/atoms is given by $A(F(x)) := \{j \in A_L : j \leqslant F(x)\}$.

For any pair F, F' of CIPs on J_L, it will be written $F' \leqq F$ whenever $F'(x) \leqslant F(x)$ for all $x \in L^{J_L}$ i.e. when F is *more inclusive* than F'.

The two basic motivating examples of our model are the following distinct and somewhat 'polar' versions of the collective identification problem.

[2] Thus, by definition, L is a finite set, and $\leqslant$ is a transitive, reflexive and antisymmetric binary relation on L.

[3] To see this, just consider a finite totally ordered set i.e. a chain $(L, \leqslant)$ with $\# L \geq 3$: the only atom of $(L, \leqslant)$ is the $\leqslant$-minimum of $L \setminus \{\wedge L\}$, while any $x \in L \setminus \{\wedge L\}$ is a join-irreducible element.

Example 1: Collective identification as collective binary self-classification with an exogenous label

That is the case the extant literature on collective identification procedures is typically focussed on: the legitimate members of a certain association or group are to be determined. Each agent provides a positive or negative assessment of all agents, and the resulting profile of assessments determines membership. Thus, the lattice is $(\mathcal{P}(N), \subseteq)$, where N is the (finite) population of agents: it is both atomistic and distributive. Indeed, in that lattice the atoms or agents are the singletons i.e. the population units themselves. Thus the standard case with set of agents N reduces to a special instance of our model with $L = \mathcal{P}(N)$, $\leqslant = \subseteq$, and $A_L = J_L \simeq N$.

Example 2: Collective identification as collective self-classification with implicit endogenous labels

In this case, population units form a partitional coalition structure i.e. partition themselves into disjoint coalitions, or blocks. Labels as such are clearly not important: they are implicitly defined by blocks themselves, and are therefore endogenously defined by the resulting partition. Thus, if N denotes the set of population units, the relevant lattice here is the lattice $(\Pi(N), \sqsubseteq)$ of partitions of N, where $\sqsubseteq$ is the *coarsening order* defined as follows: for any $\pi_1, \pi_2 \in \Pi(N)$, $\pi_1 \sqsubseteq \pi_2$ iff for any $A \in \pi_1$, and $h, k \in N$, if $\{h, k\} \subseteq A$ then there exists $B \in \pi_2$ such that $\{h, k\} \subseteq B$. This lattice is atomistic but *nondistributive*[4]: the atoms are those partitions π_{hk} of N consisting of singletons except for *an unique* two-unit block $\{h, k\}$. Therefore, the relevant agents are *pairs of distinct population units*. Hence, we have here another special instance of our model with $L = \Pi(N)$, $\leqslant = \sqsubseteq$, and $A_L = J_L \simeq (N \times N) \setminus \Delta_N$ (where $\Delta_N = \{(i, i) : i \in N\}$).

The Libertarian CIP F^l as defined on an arbitrary finite atomistic lattice establishes that the associative structure induced by an opinion profile is *the one engendered by those agents who self-certify their own qualifications -or willingness- to join*, namely

Definition 1 *(The **Libertarian** CIP F^l): for any $x \in L^{J_L}$,*

$F^l(x) = \vee \{j \in J_L : \ j \leqslant x_j \}$

[4] The partition lattice only satisfies *upper semimodularity* (a condition strictly weaker than distributivity), namely for any $a, b \in \Pi(N)$ if a is an upper cover of $a \wedge b$ then $a \vee b$ is an upper cover of b.

Hence, the libertarian rule identifies as members precisely those agents *who are components of the smallest associative structure comprising as components all those who declare themselves to qualify.* It might seem that the libertarian rule may also be described in a less cumbersome way saying it identifies as members *all those agents who declare themselves to qualify.* However, it turns out that those two descriptions are *not* equivalent whenever the relevant lattice of associative structures is *nondistributive: this is a key observation which underlies all of the ensuing analysis.*

In that connection, a basic property of (finite) distributive lattices is to be recalled, namely

Fact: (see e.g. Grätzer (1998), Monjardet (1990)). Let $\mathbf{L} = (L', \leqslant)$ be a (finite) lattice, and J^* the set of join-irreducible elements of $(L', \leqslant)$. Then, i) if $\mathbf{L}$ is distributive, then for any $x \in L'$ there exists a unique $J(x) = \{j_1, ..., j_k\} \subseteq J^*$such that $x = \vee J(x)$ and $x < \vee J'$ for any $J' \subset J(x)$; ii) $\mathbf{L}$ is distributive if and only if for any $j \in J^*$, and any $l_1, ..., l_h \in L'$, if $j < l_1 \vee ... \vee l_h$ then there exists $i \in \{1..., h\}$ such that $j \leqslant l_i$.

Thus, if $\mathbf{L}$ is (atomistic and) distributive, for any CIP $F : L^{A_L} \rightarrow L$, and any $x \in L^{A_L}$, there exists a unique (minimal or irredundant) set $\{j_1, ..., j_k\} \subseteq A_L$ such that $F(x) = j_1 \vee ... \vee j_k$ (i.e. $\{j_1, ..., j_k\}$ is an irredundant join-decomposition of $F(x)$). Hence $\{j_1, ..., j_k\} = A(F(x))$ as defined above, and *the identity $F(x) = j_1 \vee ... \vee j_k$ may be taken to denote **without any ambiguity** that CIP F at opinion profile x identifies agents $j_1, ..., j_k$ as the only legitimate members constituting the associative structure under consideration.* If, however, $\mathbf{L}$ is nondistributive then neither i) nor ii) above hold. Failure of i) entails that there may exist several distinct irredundant join-decompositions of some $l \in L$. Hence, in particular it may the case that $F(x) = j_1 \vee ... \vee j_k$ while $\{j_1, ..., j_k\} \subset A(F(x))$, hence that there exist $i \in A(F(x))$, $i \notin \{j_1, ..., j_k\}$ such that -by definition of atom- $i \nleqslant j_h$, $h = 1, ..., k$ (violating ii) as well).

To check this, just consider the partition lattice $(\Pi(\{1, 2, 3\}), \sqsubseteq)$, and its atoms π_{12}, π_{13}, π_{23}. Of course, $\pi_{12} \vee \pi_{23} = \{\{1, 2, 3\}\}$, the coarsest partition. Hence, of course, $\pi_{13} < \pi_{12} \vee \pi_{23}$, while both $\pi_{13} \nleqslant \pi_{12}$ and $\pi_{13} \nleqslant \pi_{23}$.

Concerning the libertarian CIP, the consequences of this simple fact are momentous. Indeed, consider the following properties for a CIP F:

Negative Atomic Self-Determination (NASD) : For any $x \in L^{J_L}$ and any $j \in A_L$, if $j \nleqslant x_j$ then $j \nleqslant F(x)$.

Positive Atomic Self-Determination (PASD) : For any $x \in L^{J_L}$ and any $j \in A_L$, if $j \leqslant x_j$ then $j \leqslant F(x)$.

Clearly enough, NASD establishes that any agent can effectively decline participation in the relevant association, while PASD requires that, conversely, any willing agent may join it.

Now, it is easily shown that in the binary classification problem the libertarian CIP is the only one which satisfies both NASD and PASD, whereas in the general classification problem with endogenous labels NASD and PASD are mutually *inconsistent*. Indeed, a much more general statement holds true, namely

Claim 2 *i) Let* $\mathbf{L} = (L, \leqslant)$ *be a finite* distributive *atomistic lattice. Then* F^l *is the only CIP that satisfies both NASD and PASD; ii) let* $\mathbf{L} = (L, \leqslant)$ *be a finite* nondistributive *atomistic lattice. Then no CIP* $F : L^{A_L} \rightarrow L$ *can satisfy both NASD and PASD. In particular,* F^l *satisfies PASD but violates NASD.*

Proof. i) Let $j \in A_L$, and $x \in L^{A_L}$ such that $j \nleqslant x_j$ and $j \leqslant F^l(x)$. Then, by definition of F^l, there exist $j_1, ..., j_k \in A_L$ such that $j_h \leqslant x_h$, $h = 1, ..., k$, and $j \leqslant \vee \{j_1, ..., j_k\}$. But then, by distributivity, there exists $j_i \in \{j_1, ..., j_k\}$ such that $j \leqslant j_i$, whence by definition of A_L, $j = j_i$. Therefore, $j \leqslant x_j$, a contradiction. It follows that F^l does satisfy NASD. On the other hand, let $j \in A_L$, and $x \in L^{A_L}$ such that $j \leqslant x_j$. Then, by definition of F^l, $j \leqslant F^l(x)$ and F^l also satisfies PASD.

Moreover, let $F : L^{A_L} \rightarrow L$ be a CIP that satisfies both NASD and PASD. Then, for any $j \in A_L$, and $x \in L^{A_L}$ if $j \leqslant F(x)$ then, by NASD, $j \leqslant x_j$ hence $j \leqslant F^l(x)$. Therefore, $F \leqslant F^l$ (since $\mathbf{L}$ is atomistic). Conversely, for any $j \in A_L$, and $x \in L^{A_L}$ if $j \leqslant F^l(x)$ then there exist $j_1, ..., j_k \in A_L$ such that $j_h \leqslant x_h$, $h = 1, ..., k$, and $j \leqslant \vee \{j_1, ..., j_k\}$. But then, by distributivity of $\mathbf{L}$, there exists $j_i \in \{j_1, ..., j_k\}$ such that $j \leqslant j_i$, whence $j = j_i$ and $j \leqslant x_j$. It follows that $j \leqslant F(x)$, by PASD. Thus, $F \leqslant F^l$ (since $\mathbf{L}$ is atomistic), hence $F = F^l$.

ii) By contradiction: let us assume that $\mathbf{L}$ is a finite *nondistributive* atomistic lattice, and $F : L^{A_L} \rightarrow L$ a CIP which satisfies both NASD and PASD. Since $\mathbf{L}$ is, in particular, nondistributive, there exist $j \in J_L = A_L$ and $y_1, ..., y_k \in L$ such that $j \nleqslant y_h$, $h = 1, ..., k$, and $j \leqslant \vee \{y_1, ..., y_k\}$. Moreover, since $\mathbf{L}$ is atomistic there exist $m \geq k$, and $j_1, ..., j_m \in A_L \setminus \{j\}$ such that

$j \leqslant \vee \{j_1, ..., j_m\}$. Now, consider $x \in L^{A_L}$ such that $j_h \leqslant x_h$, $h = 1, ..., m$, and $j \nleqslant x_j$. Then, by NASD, $j \nleqslant F(x)$. However, by PASD, $j_h \leqslant F(x)$, $h = 1, ..., m$, whence $j \leqslant \vee \{j_1, ..., j_m\} \leqslant F(x)$, a contradiction.

In particular, it is immediately checked that, by definition, $j \leqslant x_j$ entails $j \leqslant F^l(x)$ i.e. F^l satisfies PASD, hence it must violate NASD. ■

The foregoing facts suggest a first, straightforward characterization of the libertarian CIP in an arbitrary atomistic lattice, namely

Proposition 3 *Let* $\mathbf{L} = (L, \leqslant)$ *be a (finite) atomistic lattice, and* $F : L^{A_L} \rightarrow L$ *a CIP. Then,* $F = F^l$ *if and only if* F *is the* least inclusive *CIP that satisfies PASD.*

Proof. We know already from the previous Claim that F^l does satisfy PASD. Moreover, let $F : L^{A_L} \rightarrow L$ be a CIP that satisfies PASD. Then, for any $j \in A_L$ and $x \in L^{A_L}$ such that $j \leqslant F^l(x)$, there exist $j_1, ..., j_k \in A_L$ such that $j_h \leqslant x_h$, $h = 1, ..., k$, and $j \leqslant \vee \{j_1, ..., j_k\}$. By PASD, $j_h \leqslant F(x)$, $h = 1, ..., k$. Hence $j \leqslant \vee \{j_1, ..., j_k\} \leqslant F(x)$ i.e. $F^l \leqq F$ as claimed. ■

Notice that, conversely, the *most* inclusive CIP that satisfies PASD is the Universal Acceptance CIP $F^{\vee L}$ which invariably selects an associative structure comprising *all* agents as members[5]. Clearly, Universal Acceptance is strictly more inclusive than F^l at any $x \in L^{A_L}$ such that $j \nleqslant x_j$ for some $j \in A_L$: hence the foregoing elementary characterization is tight.

It is immediately checked that, whenever $\mathbf{L}$ is distributive as well, F^l also admits a dual characterization as the *most inclusive* CIP that satisfies NASD. Of course, in view of the foregoing Claim, such a dual characterization of F^l is bound to fail in the nondistributive case. Moreover, it has been already noticed (see Claim i) above) that in the distributive case -as opposed to the nondistributive one- F^l can also be characterized as the only CIP that jointly satisfies PASD and NASD. But then, what about the possility to lift the latter characterizations of F^l to arbitrary atomistic lattices by some suitable generalization of NASD?[6]

[5] Thus, $F^{\vee L}$ is defined as follows: for any $x \in L^{J_L}$, $F^{\vee L}(x) = \vee L$.

[6] To be sure, one might rather take the notion of a'most inclusive NASD-consistent CIP' as the starting point of the *tentative definition* of a new, dual libertarian rule F^{l^*} defined pointwise by the least upper bound of the values of those CIPs that satisfy NASD (if it exists). The main problem with this approach is that indeed the foregoing least upper

It turns out that such a lifting is indeed feasible, thanks to the following generalization of NASD.

Admission by Qualified Invitation (AQI) For any $x \in L^{A_L}$ and any $j \in A_L$, if $j \leqslant F(x)$ then there exist $j_1, ..., j_k \in A_L$ such that $j \leqslant \vee \{j_1, ..., j_k\}$ and $j_h \leqslant x_h \wedge F(x)$, $h = 1, ..., k$.

Notice that in the present setting relation $j \leqslant \vee \{j_1, ..., j_k\}$ may be interpreted as follows: 'agent j is *invited* within the relevant associative structure by agents $j_1, ..., j_k$'; such an *invitation* amounts to an arbitrary combination of explicit, formal certifications (when $j \leqslant x_{j_h}$, $h \in \{1, ..., k\}$) and implicit, tacit invitations (when $j \nleqslant x_{j_h}$, $h \in \{1, ..., k\}$) on the part of agents in $\{j_1, ..., j_k\}$.

Thus, AQI establishes that admission of an agent as member of the relevant associative structure requires either a (possibly tacit) invitation or an explicit certification on the part of *some* self-certified member(s). In general, NASD implies AQI while the converse implication does not hold. However, AQI and NASD are equivalent whenever $\mathbf{L}$ is distributive, namely

Claim 4 *i) Let* $\mathbf{L} = (L, \leqslant)$ *be a (finite) atomistic lattice, and* $F : L^{A_L} \to L$ *a CIP. Then, F satisfies NASD only if it also satisfies AQI. However, it may be the case that F does satisfy AQI while violating NASD; ii) Let* $\mathbf{L} = (L, \leqslant)$ *be a (finite) atomistic lattice. Then,* $\mathbf{L}$ *is distributive if and only if each CIP* $F : L^{A_L} \to L$ *that satisfies AQI does also satisfy NASD.*

Proof. i) Let F satisfy NASD. Therefore, if $j \in A_L$ and $j \leqslant F(x)$ then it must be the case that $j \leqslant x_j$, hence AQI is satisfied (just take $k = 1$, and $j_1 = j$). Conversely, consider the Libertarian CIP F^l : F^l trivially satisfies AQI, but fails to satisfy NASD (to check that, just consider the partition lattice $\mathbf{L} = (\Pi(\{1, 2, 3\}), \sqsubseteq)$, its atoms π_{12}, π_{13}, π_{23}, and (with a slight abuse of notation) opinion-profile $x = (x_{12}, x_{13}, x_{23})$ where $x_{12} = \pi_{12}$, $x_{13} = \pi_{13}$, $x_{23} = \{\{1\}, \{2\}, \{3\}\}$. Of course, $\pi_{12} \leqslant F^l(x)$ and $\pi_{13} \leqslant F^l(x)$. Now, $\pi_{12} \vee \pi_{13} = \{\{1, 2, 3\}\}$ and $\pi_{23} < \{\{1, 2, 3\}\}$. Thus, $F^l(x) = \{\{1, 2, 3\}\}$ whence $\pi_{23} \leqslant F^l(x)$, while $\pi_{23} \nleqslant x_{23}$).

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bound may not exist. Thus, such a 'most inclusive NASD-consistent CIP' (as opposed to 'one of many maximally inclusive NASD-consistent CIPs') is typically not a well-defined notion in the nondistributive case. The details, however, need not detain us here, and will be discussed elsewhere.

ii) Let $\mathbf{L}$ be a distributive lattice, $F : L^{A_L} \to L$ a CIP that satisfies AQI, $x \in L^{A_L}$, and $j \in A_L$. If $j \leqslant F(x)$ then, by AQI, there exists a set of atoms $\{j_1, ..., j_k\} \subseteq A_L$ such that $j_h \leqslant x_h \wedge F(x)$ for each h, $h = 1, ..., k$, and $j \leqslant \vee \{j_1, ..., j_k\}$. But then, by distributivity of $\mathbf{L}$ (see Fact ii) above), there exists $j_h \in \{j_1, ..., j_k\}$ such that $j \leqslant j_h$ i.e. $j = j_h$ (by definition of atom). Therefore, $j \leqslant x_j$ and NASD is satisfied, as claimed.

Conversely, suppose that for any CIP $F : L^{A_L} \to L$, if F satisfies AQI then it also satisfies NASD. Then, consider F^l: since it satisfies AQI, it does also satisfy NASD i.e. for any $j \in A_L$, and any $x \in L^{A_L}$, if $j \leqslant F^l(x)$ then $j \leqslant x_j$. Let us now assume that $\mathbf{L}$ is nondistributive: then, by Fact ii) above, there exist $j \in A_L, x_1, ..., x_k \in L$ such that $j < x_1 \vee ... \vee x_k$ and $j \nleqslant x_h$, $h = 1, ..., k$. Then, there exist $j_1, ..., j_m \in A_L$ such that -for each $i \in \{1, ..., m\}$- $j_i \leqslant x_h$ for some $h \in \{1, ..., k\}$, and $j < j_1 \vee ... \vee j_m$. Clearly, by construction, $j \nleqslant j_i$, $i = 1, ..., m$. Next, consider opinion-profile $x \in L^{A_L}$ such that $x_{j_h} = j_h$, $h = 1, ..., m$, and $x_i = \wedge L$ for any $i \in A_L \setminus \{j_1, ..., j_m\}$. But then, $j \leqslant j_1 \vee ... \vee j_m$, and $j \notin \{j_1, ..., j_m\}$, while by definition $F^l(x) = j_1 \vee ... \vee j_m$. Hence, $j \leqslant F^l(x)$. It follows that -by NASD- $j \leqslant x_j$, a contradiction. ■

The next elementary characterizations of the Libertarian rule based upon AQI confirm that the latter is a suitable counterpart of NASD in the general setting of arbitrary atomistic lattices.

Theorem 5 *Let $\mathbf{L} = (L, \leqslant)$ be a (finite) atomistic lattice, and $F : L^{A_L} \to L$ a CIP. Then, the following statements are equivalent: i)F is the most inclusive CIP that satisfies AQI; ii) F satisfies both PASD and AQI; iii) $F = F^l$.*

Proof. i)$\Longleftrightarrow$iii) F^l does satisfy AQI as shown above. Now, let $F : L^{A_L} \to L$ be a CIP that satisfies AQI, and $x \in L^{A_L}$, $j \in A_L$ such that $j \leqslant F(x)$. Then, by AQI, there exist $j_1, ..., j_k \in A_L$ such that $j \leqslant \vee \{j_1, ..., j_k\}$ and $j_h \leqslant x_h \wedge F(x)$, $h = 1, ..., k$. Hence, by definition of F^l, $j \leqslant F^l(x)$, i.e. $F \leqq F^l$ as required.

ii) $\Rightarrow$iii): Let $F : L^{A_L} \to L$ be a CIP that satisfies both PASD and AQI, $x \in L^{A_L}$, and $j \in A_L$. If $j \leqslant F(x)$ then by AQI there exist $j_1, ..., j_k \in A_L$ such that $j \leqslant \vee \{j_1, ..., j_k\}$ and $j_h \leqslant x_h \wedge F(x)$, $h = 1, ..., k$. Therefore, $j \leqslant F^l(x)$, by definition of F^l. Moreover, if $j \leqslant F^l(x)$ then there exist $j_1, ..., j_k \in A_L$

such that $j_h \leqslant x_h$, $h = 1, ..., k$, and $j \leqslant \vee \{j_1, ..., j_k\}$. Therefore, by PASD, $j_h \leqslant F(x)$, $h = 1, ..., k$, hence $j \leqslant \vee \{j_1, ..., j_k\} \leqslant F(x)$.

iii)⇒ii): The Libertarian rule F^l clearly satisfies PASD (see the previous Claim). F^l also satisfies AQI: just consider that, for any $j \in A_L$, $x \in L^{A_L}$ such that $j \leqslant F^l(x)$ it must be the case that $j \in A_L$. ■

Notice that the foregoing characterizations are tight, since there exist CIPs which satisfy AQI and are strictly *less* inclusive than F^L, while PASD and AQI are mutually independent axioms. To check this, consider the Universal Acceptance CIP $F^{\vee L}$ and the Universal Rejection CIP $F^{\wedge L}$: $F^{\vee L}$ satisfies PASD but not AQI, while $F^{\wedge L}$ satisfies AQI but not PASD, and is clearly less inclusive than F^l.

2.1 Non-manipulability of the Libertarian rule in simple environments

Since CIPs are strategic game forms having opinions as strategies, it makes sense to enquire about their manipulability or, more generally, their solvability with respect to some suitable noncooperative or cooperative game-theoretic solution concepts. We only need to specify the set of admissible preferences over outcomes of each agent. Here I shall focus on a very simple set of admissible preference profiles, leaving a more general, full-fledged analysis as a topic for further research.

Consider the most elementary case of self-oriented preferences, where each agent $i \in N$ only cares about her own status with respect to association. Then, in the basic case of (binary) identification with exogenous labels agent i will partition the outcome set L into *two* equivalence classes, namely the two sets of best and worst outcomes, characterized by consistency with her own preferred and dispreferred membership status, respectively. This case motivates the general notion of a *simple* environment as made precise by the following

Definition 6 *Let $\mathbf{L} = (L, \leqslant)$ be a finite atomistic lattice, A_L the set of its atoms, and $x = (x_j)_{j \in A_L} \in L^{A_L}$. An A_L-profile $(\succcurlyeq_i)_{i \in A_L}$ of binary (preference) relations on L^{A_L} is* simple *w.r.t. x -written $(\succcurlyeq_i)_{i \in A_L} \in \mathcal{S}^{A_L}(x)$- iff for any $i \in A_L$, and any $y, z \in L^{A_L}$: $y \succcurlyeq_i z$ if and only if either $[j \leqslant y$ and $j \leqslant x_j]$ or $[j \nleqslant y$ and $j \nleqslant x_j]$. The* simple *environment (on $(\mathbf{L}, A_L)$) consists* *of the set* $\mathcal{S}^{A_L} = \cup_{x \in L^{A_L}} \mathcal{S}^{A_L}(x)$ *of all preference* A_L*-profiles on* L^{A_L} *that are simple w.r.t. some* $x \in L^{A_L}$.

The significance of the simple environment rests both on its remarkable tractability and on the fact that it apparently leaves as little scope as possible for strategic manouvering and manipulation. The relevant notion of (coalitional) non-manipulability or strategy-proofness is a straightforward adaptation of the standard concept for voting mechanisms, namely

Definition 7 *Let* $\mathbf{L} = (L, \leqslant)$ *be a finite atomistic lattice, and* $F : L^{A_L} \rightarrow L$ *a CIP.* F *is* coalitionally strategy-proof *on the simple domain* $\mathcal{S}^{A_L}$ *iff for any* $x \in L^{A_L}$, A_L*-profile* $(\succcurlyeq_i)_{i \in A^L} \in \mathcal{S}^{A_L}(x)$, $S \subseteq A_L$, $z_{N \setminus S} \in L^{N \setminus S}$, *and* $y_S \in L^S$, *there exists* $i \in S$ *such that*
$F(x_S, z_{N \setminus S}) \succcurlyeq_i F(y_S, z_{N \setminus S})$.

It turns out that at least on a very restricted domain such as the simple environment, the Libertarian rule studied in this paper is indeed coalitionally strategy-proof, *no matter whether the underlying (finite, atomistic) lattice is distributive or not:*

Proposition 8 *Let* $\mathbf{L} = (L, \leqslant)$ *be a finite atomistic lattice. Then the Libertarian CIP* $F^l : L^{A_L} \rightarrow L$ *is coalitionally strategy-proof on the simple domain* $\mathcal{S}^{A_L}$.

Proof. Let $\mathbf{L}$ be an atomistic lattice, and $A_L = \{j_1, .., j_k\}$. Then, consider $x = (x_j)_{j \in A_L} \in L^{A_L}$, $(\succcurlyeq_i)_{i \in A^L} \in \mathcal{S}^{A_L}(x)$.

Suppose there exist $S \subseteq A_L$, $y_S \in L^S$ and $z_{N \setminus S} \in L^{N \setminus S}$ such that for each $i \in S$

$$F^l(y_S, z_{N \setminus S}) \succ_i F^l(x_S, z_{N \setminus S}).$$

Then, by definition of $\mathcal{S}^{A_L}(x)$, it must be the case that for any $i \in S$ either

(i) $i \leqslant x_i$, $i \nleqslant F^l(x_S, z_{N \setminus S})$ and $i \leqslant F^l(y_S, z_{N \setminus S})$

or

(ii) $i \nleqslant x_i$, $i \leqslant F^l(x_S, z_{N \setminus S})$ and $i \nleqslant F^l(y_S, z_{N \setminus S})$.

However, by definition of F^l, $i \leqslant x_i$ and $i \in S$ entail $i \leqslant F^l(x_S, z_{N \setminus S})$ contradicting (i). Hence, (ii) holds for each $i \in S$. Now, posit $u = (x_S, z_{N \setminus S})$: for any $i \in S$, since $i \nleqslant x_i$ and $i \leqslant F^l(x_S, z_{N \setminus S})$, there exists $J_i = \{j_1, ..., j_{k_i}\} \subseteq A_L$ such that $j_h \leqslant u_h$, $h = 1, ..., k_i$, and $i \leqslant \vee \{j_1, ..., j_{k_i}\}$, by definition of

F^l. But then, for any such profile $(J_i)_{i\in S}$, $S\cap(\cup_{i\in S}J_i)=\emptyset$ whence, for any $i\in S$, $i\nleqslant x_i$ and $i\leqslant F^l(x_S,z_{N\setminus S})$ entail $i\leqslant F^l(y_S,z_{N\setminus S})$ for any $i\in S$, a contradiction again. ■

Clearly, coalitional strategy-proofness of F^l (on the simple domain) amounts to say that at any $x\in L^{A_L}$, x is a coalitionally dominant strategy equilibrium[7] of the game in strategic form $G(F^l,(\succcurlyeq_i(x))_{i\in A_L}):=(A_L,L,(L_i=L)_{i\in A_L},F^l,(\succcurlyeq_i(x))_{i\in A_L})$. But notice that a coalitionally dominant strategy equilibrium of a game is also a strong equilibrium and a fortiori a coalitional equilibrium with threats[8] of the same game.

Therefore, for any atomistic lattice the Libertarian rule enjoys both non-cooperative and cooperative stability at least on the simple domain[9].

3 Related literature

As mentioned in the Introduction, a few axiomatic characterizations of the libertarian rule have been provided in the literature on collective identification with *exogenous* labels (both binary and nonbinary). In this section we shall briefly review them with a view to assess whether they can be lifted into the general case of arbitrary atomistic lattices, in order to cover the case of collective identification with *endogenous labelling.* As it turns out, none of

[7]An A_L-profile $y\in L^{A_L}$ is a *coalitionally dominant strategy equilibrium* of $G(F,(\succcurlyeq_i)_{i\in A_L}F)$ iff for any $T\subseteq A_L$, $u_{N\setminus T}\in L^{N\setminus T}$ and $z_T\in L^T$ there exists $i\in T$ such that
$F(y_T,u_{N\setminus T})\succcurlyeq_i F(z_T,u_{N\setminus T})$.

[8]An A_L-profile $y\in L^{A_L}$ is a *coalitional equilibrium with threats* of $G(F,(\succcurlyeq_i)_{i\in A_L}F)$ iff for any $T\subseteq A_L$ and $z_T\in L^T$ there exists $w_{N\setminus T}\in L^{N\setminus T}$ and $i\in T$ such that
$F(y)\succcurlyeq_i F(w_{N\setminus T},z_T)$.
Moreover, a coalitional equilibrium with threats y of $G(F,(\succcurlyeq_i)_{i\in A_L})$ is a *strong equilibrium* of $G(F,(\succcurlyeq_i)_{i\in A_L})$ iff in particular for any $T\subseteq A_L$ and $z_T\in L^T$ there exists $i\in T$ such that
$F(y)\succcurlyeq_i F(y_{N\setminus T},z_T)$.
Of course, any strong equilibrium of F at $(\succcurlyeq_i)_{i\in A_L}$ is a coalitional equilibrium with threats of F at $(\succcurlyeq_i)_{i\in A_L}$, but not vice versa. Also notice that coalitional equilibrium with threats is the strategic counterpart of the *core*, namely any $l\in L$ is a *core outcome* of $G(F,(\succcurlyeq_i)_{i\in A_L})$ iff there exists a coalitional equilibrium with threats y of F at $(\succcurlyeq_i)_{i\in A_L}$ such that $l=F(y)$.

[9]A strategic game form G is solvable (or stable) with respect to a certain solution concept on a certain domain D of preference profiles, if at each preference profile $\succcurlyeq$ in D the game $(G,\succcurlyeq)$ has a nonempty set of solutions.

the known characterizations of the libertarian CIP works within an arbitrary atomistic lattice: in the process, we shall obtain generalizations of some of the foregoing characterizations, showing that they hold *if and only if* the underlying lattice of feasible associative structures is *distributive*.

Indeed, most known characterizations of the libertarian CIP for binary classification problems rest on the basic requirement that membership of any agent should only depend on the assessment of *her own* credentials, as established by the most straightforward adaptation of *arrowian Independence* to collective identification problems, namely

Independence (IND): For any $x, x' \in L^{A_L}$ and any $j \in A_L$ if [for all $i \in A_L$: $j \leqslant x_i$ iff $j \leqslant x'_i$] then [$j \leqslant F(x)$ iff $j \leqslant F(x')$].

Samet and Schmeidler (2003) provide two characterizations of the Libertarian rule as defined on the lattice $(\mathcal{P}(N), \supseteq)$, relying on Independence and the first three properties (the first, the second and the fourth property, respectively) of the following list:

Nondegeneracy (NDG): For any $j \in A_L$ there exist $x, x' \in L^{A_L}$ such that $j \leqslant F(x)$ and $j \nleqslant F(x')$.

Clearly, Nondegeneracy is a mild requirement ensuring that for any agent there exist both opinion profiles resulting in her inclusion in the associative structure and opinion profiles mandating her exclusion from the latter. Thus, Nondegeneracy rules out trivial constant rules and guarantees that each agent may be or may be not part of the relevant association, depending on the prevailing opinion profile.

Monotonicity (MON): For any $x, x' \in L^{A_L}$, if $x_j \leqslant x'_j$ for each $j \in A_L$ then $F(x) \leqslant F(x')$.

Monotonicity is also quite standard: it simply establishes that shifts to opinion profiles acknowledging larger sets of qualified agents cannot result in reduced memberships.

Collective Self-Determination (CSD): For any $x, x' \in L^{A_L}$ if [$j \leqslant x_i$ iff $i \leqslant x'_j$ for any $i, j \in A_L$] then $F(x) = F(x')$.

Collective Self-Determination is a symmetry condition for members which rules out cooptation practices: reversing the roles of 'certifiers' and 'certified' agents should not affect membership.

Exclusivity of Self-Determination (ESD): For any $x, x' \in L^{A_L}$ and any $i, j \in A_L$ if [(($j \leqslant x_i$ and $j \nleqslant x'_i$) or ($j \nleqslant x_i$ and $j \leqslant x'_i$)) iff ($i \nleqslant F(x)$ and $j \leqslant F(x)$)] then $F(x) = F(x')$.

Thus, Exclusivity of Self-Determination decrees the irrelevance of the opinion of non-members: changes in the opinion of the latter should not affect membership.

Notice that the Libertarian CIP satisfies NDG, MON, CSD and ESD in any (finite) atomistic lattice, namely

Proposition 9 *Let $\mathbf{L} = (L, \leqslant)$ be a finite atomistic lattice. Then the Libertarian CIP $F^l : L^{A_L} \to L$ does satisfy NDG, MON, CSD and ESD.*

Proof. To see that F^l satisfies NDG just consider profiles $x = (x_j = \vee L)_{j \in A_L}$ and $y = (y_j = \wedge L)_{j \in A_L}$: clearly, for any $i \in A_L$, $i \leqslant F^l(x)$ and $i \nleqslant F^l(y)$. That F^l satisfies MON is also straightforward: if $i \leqslant F^l(x)$ and $x_j \leqslant x'_j$ for any $j \in A_L$ then, in particular $j \leqslant x_j$ entails $j \leqslant x'_j$, hence, by definition of F^l, $i \leqslant F^l(x')$ as well. Concerning CSD, notice that by definition of x and x', $i \leqslant x_i$ iff $i \leqslant x'_i$. Therefore $j \leqslant \vee \{j_1, ..., j_k\}$ with $\{j_1, ..., j_k\} \subseteq \{i : i \leqslant x_i\}$ iff $j \leqslant \vee \{j_1, ..., j_k\}$ with $\{j_1, ..., j_k\} \subseteq \{i : i \leqslant x'_i\}$ whence $F^l(x) = F^l(x')$.

Finally, for any $x, x' \in L^{A_L}$ and any $i, j \in A_L$ if $[((j \leqslant x_i$ and $j \nleqslant x'_i)$ or $(j \nleqslant x_i$ and $j \leqslant x'_i))$ iff $(i \nleqslant F^l(x)$ and $j \leqslant F^l(x))]$ then by definition of F^l it must be the case that $x_i \neq x'_i$ entails $i \nleqslant x_i$ hence in particular $j \leqslant F^l(x)$ entails $j \leqslant F^l(x')$. Conversely, $j \leqslant F^l(x')$ entails $j \leqslant \vee \{j_1, ..., j_k\}$ with $\{j_1, ..., j_k\} \subseteq \{i : i \leqslant x'_i\}$ hence, by definition of x', $\{i : i \leqslant x'_i\} \subseteq \{i : i \leqslant x_i\}$. Therefore, $j \leqslant F^l(x)$ as well. It follows that ESD is also satisfied by F^l. ■

However, it turns out that Independence is satisfied by the Libertarian CIP F^l if $\mathbf{L}$ is distributive, but *not* in the general case, as made precise by the following

Proposition 10 *Let $\mathbf{L} = (L, \leqslant)$ be a finite atomistic lattice. Then the Libertarian CIP $F^l : L^{A_L} \to L$ satisfies IND iff $\mathbf{L}$ is distributive.*

Proof. Let $\mathbf{L} = (L, \leqslant)$ be a finite *distributive* atomistic lattice, $j \in A_L$ and $x, x' \in L^{A_L}$ such that for all $i \in A_L$, $j \leqslant x_i$ iff $j \leqslant x'_i$. If $j \leqslant F^l(x)$ there must exist $j_1, ..., j_k \in A_L$ such that $j_h \leqslant x_h$, $h = 1, ..., k$, and $j \leqslant \vee \{j_1, ..., j_k\}$. By distributivity, there exists $h \in \{1, ..., k\}$ such that $j \leqslant j_h$ i.e. $j = j_h$ since $j \in A_L$. Therefore, $j \leqslant x_j$ hence, by assumption, $j \leqslant x'_j$: it follows that $j \leqslant F^l(x')$. By a similar argument it is easily checked that $j \leqslant F^l(x')$. Thus F^l does indeed satisfy IND.

Conversely, let us suppose that F^l satisfies IND. If $\mathbf{L}$ is *not* distributive then, by Fact (ii) above there exist $j \in A_L$ and $x_1, ..., x_k \in L$ such that $j_h \nleqslant x_h$, $h = 1, ..., k$, and $j \leqslant \vee \{x_1, ..., x_k\}$. Thus, again, there also exist

$j_1,...,j_m \in A_L$ such that -for each $i \in \{1,...,m\}$- $j_i \leqslant x_h$ for some $h \in \{1,...,k\}$, and $j < j_1 \vee ... \vee j_m$. Clearly, by construction, $j \nleqslant j_i$, $i = 1,...,m$.

Now, consider opinion-profiles $x = (x_i)_{i \in A_L}, x' = (x'_i)_{i \in A_L} \in L^{A_L}$ such that $x_j = x'_j = \wedge L$ while for any $i \in A_L \setminus \{j\}$, $x_i = \wedge L$ and $x'_i = i$. Clearly $\{i \in A_L : j \leqslant x_i\}$=$\{i \in A_L : j \leqslant x'_i\}$=$\emptyset$ and $j \nleqslant F^l(x)$ but $j \leqslant F^l(x')$ $\{i \in A_L : j \leqslant x_i\}$, so IND is violated: contradiction. ■

Thus, under the Libertarian CIP in an *arbitrary* atomistic lattice, the assessment of the qualification of any agent may well also depend on the assessment of the qualifications of *other* agents. As a consequence, we have the following generalization of the characterization(s) of the Libertarian CIP due to Samet and Schmeidler (2003).

Proposition 11 *Let* $\mathbf{L} = (L, \leqslant)$ *be a finite atomistic lattice, and* $F : L^{A_L} \rightarrow L$ *a CIP that satisfies [NDG, MON, CSD] or [NDG, MON, ESD] . Then, the following statements are equivalent:*

i) $\mathbf{L}$ *is distributive; ii) If* F *satisfies IND then* $F = F^l$.

Proof. i)$\Longrightarrow$ii) Suppose $\mathbf{L}$ is distributive. Then by a straightforward adaptation of the proof of Theorem 3 (Theorem 4, respectively) of Samet and Schmeidler (2003) to the case of a general distributive lattice, and by Propositions 9 and 10 above, F^l uniquely satisfies IND, NDG, MON, CSD (IND, NDG, MON, ESD, respectively). Therefore, if F satisfies IND as well then $F = F^l$.

ii)$\Longrightarrow$i) Suppose F does indeed satisfy IND. Then, by assumption, it must be the case that $F = F^l$. Therefore, F^l does also satisfy IND. But then, by Proposition 10 above, $\mathbf{L}$ is distributive. ■

Another interesting characterization of the Libertarian CIP in the lattice $(\mathcal{P}(N), \supseteq)$ - due to Kasher and Rubinstein (1997) and further refined by Sung and Dimitrov (2005)- relies on the following properties

Conditional Independence (CI): For any $j \in A_L$ and $x, x' \in L^{A_L}$, if [i) $i \leqslant F(x)$ iff $i \leqslant F(x')$ for any $i \in A_L \setminus \{j\}$, and ii) $j \leqslant x_h$ iff $j \leqslant x'_h$ for all $h \in A_L$] then $j \leqslant F(x)$ iff $j \leqslant F(x')$.

Thus, CI is a very weak version of Independence (see section 3 below): it requires that membership of any agent should only depend on the assessment of her qualifications *and* on assignment of memberships to other agents.[10]

Positive Opinion Responsiveness (POR): For any $x \in L^{A_L}$, if there exists $j \in A_L$ such that $j \leqslant x_j$ then there also exists $i \in A_L$ such that $i \leqslant F(x)$.

Clearly, POR requires that if some agent self-certifies her qualifications to join the given associative structure then at least one agent (not necessarily the same) should be admitted into the latter. POR amounts to a generalized version of the 'positive' part of the so-called 'Liberal principle' due to Kasher and Rubinstein (1997)).

Negative Opinion Responsiveness (NOR): For any $x \in L^{A_L}$, if there exists $j \in A_L$ such that $j \nleqslant x_j$ then there also exists $i \in A_L$ such that $i \nleqslant F(x)$.

Hence, NOR is the 'negative' part of Kasher-Rubinstein's 'Liberal principle' saying that if some agent denies her own qualifications then the associative structure should *not* include each agent.

Horizontal Symmetry (HS): For any $i, j \in A_L$ and $x \in L^{A_L}$, $i \leqslant F(x)$ iff $j \leqslant F(x)$ whenever [i) $i \leqslant x_i$ iff $j \leqslant x_j$, ii) $i \leqslant x_j$ iff $j \leqslant x_i$, iii) $i \leqslant x_h$ iff $j \leqslant x_h$ and $h \leqslant x_i$ iff $h \leqslant x_j$, for any $h \in A_L \setminus \{i, j\}$].

HS essentially requires equal treatment for any two agents whose qualifications are assessed identically by themselves and by all other agents: it is a generalized version of the so called 'Symmetry' property introduced by Kasher and Rubinstein (1997) (see also Sung and Dimitrov (2005)).

As it happens, the Libertarian CIP F^l satisfies CI and POR in any atomistic lattice; however, NOR and HS are only satisfied by F^l when the underlying lattice is distributive, namely

Proposition 12 *Let* $\mathbf{L} = (L, \leqslant)$ *be a finite atomistic lattice. Then, i)* $F^l : L^{A_L} \to L$ *satisfies CI and POR; ii) if* $\mathbf{L}$ *is distributive then* $F^l : L^{A_L} \to L$ *does satisfy HS; iii)* $F^l : L^{A_L} \to L$ *satisfies NOR iff* $\mathbf{L}$ *is distributive.*

Proof. i) Let $j \in A_L$ and $x, x' \in L^{A_L}$, such that [$i \leqslant F^l(x)$ iff $i \leqslant F^l(x')$ for any $i \in A_L \setminus \{j\}$] and [$j \leqslant x_h$ iff $j \leqslant x'_h$ for all $h \in A_L$]. Now, suppose $j \leqslant F^l(x)$. Then, there exist $j_1, ..., j_k \in A_L$ such that $j_h \leqslant x_h$, $h = 1, ..., k$, and

[10] CI is also an adaptation of a property introduced by Kasher and Rubinstein (1997) under the misleading label 'Independence'.

$j \leqslant \vee \{j_1, ..., j_k\}$. Hence, by definition, $j_h \leqslant F^l(x)$, $h = 1, ..., k$, and therefore $j_h \leqslant F^l(x')$ for any $j_h \in \{j_1, ..., j_k\}$. It follows that $j \leqslant \vee \{j_1, ..., j_k\} \leqslant F^l(x')$. By a similar argument $j \leqslant F^l(x')$ entails $j \leqslant F^l(x)$. Thus, F^l satisfies CI.

Moreover, for any $x \in L^{A_L}$, if there exists $j \in A_L$ such that $j \leqslant x_j$ then by definition $j \leqslant F^l(x)$, hence POR is also satisfied by F^l.

ii) Suppose $\mathbf{L}$ is distributive, and let $x \in L^{A_L}$, $i, j \in A_L$ be such that a) $i \leqslant x_i$ iff $j \leqslant x_j$, b) $i \leqslant x_j$ iff $j \leqslant x_i$, c) $i \leqslant x_h$ iff $j \leqslant x_h$ and $h \leqslant x_i$ iff $h \leqslant x_j$, for any $h \in A_L \setminus \{i, j\}$. Then, $i \leqslant F^l(x)$ implies that there exist $j_1, ..., j_k \in A_L$ such that $j_h \leqslant x_h$, $h = 1, ..., k$, and $i \leqslant \vee \{j_1, ..., j_k\}$. By distributivity, there exists $j_h \in \{j_1, ..., j_k\}$ such that $i \leqslant j_h$ hence indeed $i = j_h$. Therefore, $i \leqslant x_i$, whence by assumption a) $j \leqslant x_j$. It follows that $j \leqslant F^l(x)$ as well, as required by HS.

iii) Let us first assume that $F^l : L^{A_L} \to L$ satisfies NOR and $\mathbf{L}$ is *not* distributive. Then, by Fact (ii) above, there exist $j \in A_L$ and $x_1, ..., x_k \in L$ such that $j_h \nleqslant x_h$, $h = 1, ..., k$, and $j \leqslant \vee \{x_1, ..., x_k\}$. Thus, again, there also exist $j_1, ..., j_m \in A_L$ such that -for each $i \in \{1, ..., m\}$- $j_i \leqslant x_h$ for some $h \in \{1, ..., k\}$, and $j < j_1 \vee ... \vee j_m$. Clearly, by construction, $j \nleqslant j_i$, $i = 1, ..., m$.

Now, consider opinion-profiles $x = (x_i)_{i \in A_L} \in L^{A_L}$ such that $x_j = \wedge L$ while for any $i \in A_L \setminus \{j\}$, $x_i = \vee L$. Then in particular $i \leqslant F^l(x)$ for *any* $i \in A_L$ (j included), while $j \nleqslant x_j$, which contradicts NOR. Conversely, if $\mathbf{L}$ *is* distributive and $x \in L^{A_L}$is such that there exists $j \in A_L$ with $j \nleqslant x_j$ then it can be shown that $j \nleqslant F^l(x)$. Indeed, suppose that $j \leqslant F^l(x)$. Then, by definition of F^l, there exist $j_1, ..., j_k \in A_L$ such that $j_h \leqslant x_h$, $h = 1, ..., k$, and $j \leqslant \vee \{j_1, ..., j_k\}$. Since $j \nleqslant x_j$, $j \notin \{j_1, ..., j_k\}$ for any such set, a contradiction in view of Fact (ii) above and distributivity of $\mathbf{L}$. ■

Remark 13 *Notice that if* $\mathbf{L}$ *is* not *distributive, then* F^l *may not satisfy HS. To check this, consider the partition lattice* $\mathbf{L} = (\Pi(\{1, 2, 3, 4, 5\}), \sqsubseteq)$, *and (with a slight abuse of notation) opinion-profile* x *such that* $x_{12} = \pi_{12}$, $x_{13} = \pi_{13}$, $x_{hk} = \{\{1\}, \{2\}, \{3\}\}$ *for any other atom (as indexed by hk). Of course,* $\pi_{23} \nleqslant x_{23}$ *and* $\pi_{45} \nleqslant x_{45}$, $\pi_{23} \nleqslant x_{45}$ *and* $\pi_{45} \nleqslant x_{23}$, *and for any* $\pi_{hk} \in A_L \setminus \{\pi_{23}, \pi_{45}\}$, *both [*$\pi_{23} \nleqslant x_{hk}$ *and* $\pi_{45} \nleqslant x_{hk}$*] and [*$\pi_{hk} \nleqslant x_{23}$ *and* $\pi_{hk} \nleqslant x_{45}$*]. Nevertheless, since* $F^l(x) = \pi_{12} \vee \pi_{13}$, *it follows that* $\pi_{23} \leqslant F^l(x)$ *while* $\pi_{45} \nleqslant F^l(x)$.

As a consequence of Proposition 11 we can establish the following generalization of the characterization result for the Libertarian rule due to Kasher and Rubinstein (1997), and Sung and Dimitrov (2005).

Proposition 14 *Let* $\mathbf{L} = (L, \leqslant)$ *be a finite atomistic lattice, and* $F : L^{A_L} \to L$ *a CIP that satisfies CI, POR and HS. Then, the following statements are equivalent:*

i) $\mathbf{L}$ *is distributive; ii) If* F *satisfies NOR then* $F = F^l$*.*

Proof. i)$\Longrightarrow$ii) Suppose $\mathbf{L}$ is distributive. Then by a straightforward adaptation of the proof of Theorem 2 of Sung and Dimitrov (2005) to the case of a general distributive lattice, and by Propositions 11 above, F^l uniquely satisfies CI, POR, NOR and HS. Therefore, if F satisfies NOR as well then $F = F^l$.

ii)$\Longrightarrow$i) Suppose F does indeed satisfy NOR. Then, by assumption, it must be the case that $F = F^l$. Therefore, F^l does also satisfy NOR. But then, by Proposition 10 above, $\mathbf{L}$ is distributive. ■

Remark 15 *Other characterizations of the Libertarian rule have been provided for the distributive. In particular, Houy (2007) has two characterizations of* F^l *that rely on Independence, while Miller (2006) provides a characterization that rests on* Join-Separability *(i.e. for any* $x, y \in L^{A_L}$, $F(x \vee y) = F(x) \vee F(y)$*) which again may be not satisfied by* F^l *if the underlying lattice is nondistributive. Thus, it transpires that virtually all known characterizations of the Libertarian Rule provided in the extant literature are not suitable for the general (atomistic) case.*

4 Concluding remarks

It has been shown that the Libertarian Identification Rule F^l can be properly defined in any atomistic lattice and thus applied in both binary classification models with exogenous intensional labels and in general classification problems with endogenous extensional labels. However, not all of the basic properties of F^l in the special (boolean) distributive case of binary classifications can be safely lifted to that general atomistic environment. As a consequence, only some of the known characterizations of the Libertarian Rule can be adapted to the latter. By contrast, (coalitional) strategy-proofness of F^l on simple domains turns out to hold for any finite atomistic lattice of associative structures.

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