Winter School IT4 January 2009 From Unidimensional to Multidimensional Measurement of Welfare, Inequality and Well-Being

Claudio Zoli University of Verona, claudio.zoli@univr.it

January 13, 2009

Aims of the talk

- Overview of selected issues underlying the theory of measurement of inequality, welfare, poverty and well being.

1. **Unidimensional:** Individuals/households are homogeneous in all ethically relevant characteristics except consumption or income.
2. **Multidimensional:** heterogeneous individuals exhibiting differences in a number of "characteristics" (transferable and not transferable) e.g. income, health, housing, bundles of goods, education, household size, level of needs.....

Aims of the talk

- Overview of selected issues underlying the theory of measurement of inequality, welfare, poverty and well being.
- Two broad perspectives:

1. **Unidimensional:** Individuals/households are homogeneous in all ethically relevant characteristics except consumption or income.
2. **Multidimensional:** heterogeneous individuals exhibiting differences in a number of "characteristics" (transferable and not transferable) e.g. income, health, housing, bundles of goods, education, household size, level of needs.....

Aims of the talk 2

- A number of interrelated perspectives of evaluation:

1. Inequality (focus on dispersions across agents),
2. Welfare (taking into account also the size of the cake)
3. Poverty (focussing on deprived agents, size and dispersion matters but only focussing on those deprived)
4. Well being: multidimensional perspective focussing on size and dispersion.

Aims of the talk 2

- A number of interrelated perspectives of evaluation:

1. Inequality (focus on dispersions across agents),
2. Welfare (taking into account also the size of the cake)
3. Poverty (focussing on deprived agents, size and dispersion matters but only focussing on those deprived)
4. Well being: multidimensional perspective focussing on size and dispersion.

- Evaluations: complete rankings (i.e. indices) or partial rankings?

Our Concern

- To provide some intuitions on the interrelations between the various concepts in the unidimensional case and then move to the MORE INTERESTING Multidimensional case........

Unidimensional/Multidimensional set up.

- n homogeneous individuals $i = 1, 2, ..., n \geq 2$
- $d \geq 1$ characteristics, goods, attributes, attainments (e.g. income) $j = 1, 2, 3, ..., d$
- Distribution $X \in \mathbb{R}_+^{n \times d}$

$$X = \begin{bmatrix} x_{11} & \cdots & \cdots & x_{1(d-1)} & x_{1d} \\ x_{21} & x_{22} & \cdots & \cdots & x_{2d} \\ \vdots & \cdot & x_{ij} & & \vdots \\ x_{n1} & x_{n2} & \cdots & \cdots & x_{nd} \end{bmatrix}$$

- $\mathbf{x}_{.j} \in \mathbb{R}_+^n$ distribution of attribute j across all individuals, (column)
- $\mathbf{x}_{i.} \in \mathbb{R}_+^d$ distribution of all the attributes for individual i, (row)

Set up.

- $F_X(\mathbf{x})$ or $F(x)$ c.d.f.
- $\mu(\mathbf{x}_{.j}) = \sum_{i=1}^{n} x_{ij}/n$ average of distribution $\mathbf{x}_{.j}$ of attribute j (e.g. income)
- $\hat{\mathbf{x}}_{.j}$ ordered distribution of attribute j:
  $\hat{x}_{(1)j} \leq \hat{x}_{(2)j} \leq ..\hat{x}_{(i)j}. \leq \hat{x}_{(n)j}$
- $I(X) : \mathbb{R}_+^{n \times d} \rightarrow \mathbb{R}$ Inequality index,
- $W(X) : \mathbb{R}_+^{n \times d} \rightarrow \mathbb{R}$ Social Evaluation Function (SEF)

The Unidimensional case

Definition (Cumulative Distribution Function)

$F : \mathbb{R}_+ \to [0,1]$ Function $F(x)$ plotting the proportion of income units within the population with income at most x.

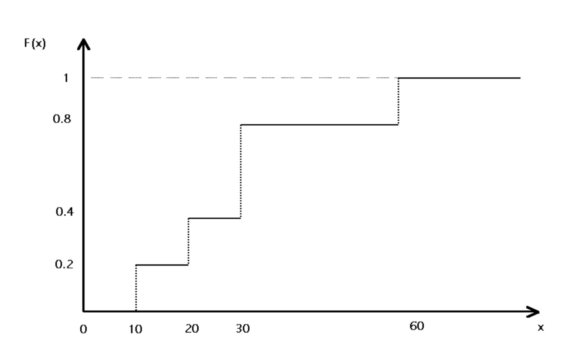


Figure: C.d.f. $F_{\mathbf{x}}(x)$ for $\mathbf{x} = (10, 20, 30, 30, 60)$

Definition (Inverse Distribution Function)

$F^{-1} : [0, 1] \rightarrow \mathbb{R}_+$. Function $F^{-1}(p) = \inf\{x \in \mathbb{R}_+ : F(x) \geq p\}$. plotting the income level corresponding to the p^{th} quantile of the population once incomes are ranked in ascending order:

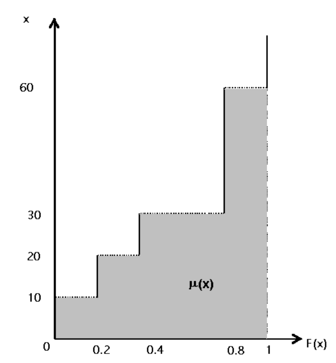


Figure: Inv.d.f. $F_{\mathbf{x}}^{-1}(p)$ for $\mathbf{x} = (10, 20, 30, 30, 60)$

Definition

Additively decomposable order $\geqslant_{\mathcal{U}}$

$$
\begin{aligned}
X &\geqslant {}_{\mathcal{U}} Y \\
&\Leftrightarrow \\
W_u(X) &= \int_{\mathbb{R}} u(x) dF_X(x) \geq \int_{\mathbb{R}} u(x) dF_Y(x) = W_u(Y) \ \forall u \in \mathcal{U}
\end{aligned}
$$

$\frac{1}{n} \sum_{i=1}^{n} u(x_i) \geq \frac{1}{n} \sum_{i=1}^{n} u(y_i) \quad \forall u \in \mathcal{U}$

The key property is Independence:

Definition (Independence)

$W_u(X, Z) \geq W_u(Y, Z)$ if and only if $W_u(X) \geq W_u(Y)$.

Definition

Rank dependent (dual) order $\geqslant_{\mathcal{V}}$

$$\begin{aligned} X &\geqslant {}_{\mathcal{V}} Y \\ &\Leftrightarrow \\ W_v(X) &= \int_0^1 v(p) \cdot F_X^{-1}(p)dp \\ &\geq \int_0^1 v(p) \cdot F_Y^{-1}(p)dp = W_v(Y) \ \forall v \in \mathcal{V} \end{aligned}$$

$\frac{1}{n}\sum_{i=1}^n v_i \cdot \hat{x}_{(i)} \geq \frac{1}{n}\sum_{i=1}^n v_i \cdot \hat{y}_{(i)} \ \forall v \in \mathcal{V}$ where $v_i \geq 0; \hat{x}_{(i)} \leq \hat{x}_{(i+1)}$

The key property is Comonotonic Independence:

Definition (Comonotonic Independence)

$W_v(X+Z) \geq W_v(Y+Z)$ if and only if $W_v(X) \geq W_v(Y)$.

The criteria $\geqslant_{\mathcal{U}}$ $\geqslant_{\mathcal{V}}$are partial orders

The most common tools applied in inequality analysis to compare income distributions are indeed the partial orders induced by the stochastic dominance conditions (direct and inverse).

Definition (Lorenz Dominance)

Define the **Lorenz curve** for X:

$$L_X(p) := \int_0^p \frac{F_X^{-1}(t)}{\mu(X)} dt.$$

Income profile X *Lorenz dominates* income profile Y, $X \succcurlyeq_L Y$, if and only if

$$L_X(p) \geq L_Y(p) \text{ for all } p \in [0, 1].$$

$$L_x(i/n) = \frac{\sum_{j=1}^{i} \hat{x}_{(j)}}{\sum_{j=1}^{n} x_j} \qquad \text{where } \hat{x}_{(i)} \leq \hat{x}_{(i+1)}$$

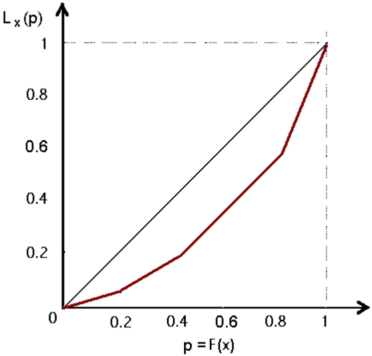


Figure: Lorenz curve for $\mathbf{x} = (10, 20, 30, 30, 60)$

Note that the Lorenz curve is obtained integrating the graph of the inverse distribution function an dividing by the average income.

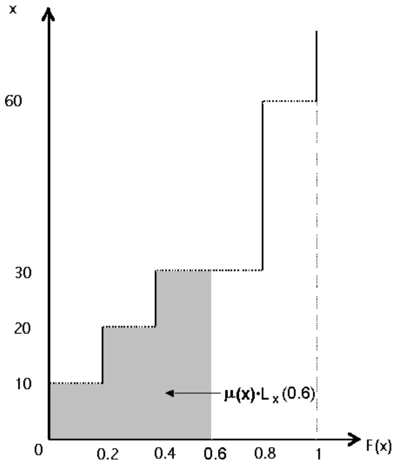


Figure: Lorenz curve derived from inverse distribution.

Definition (Generalized Lorenz Dominance)

Define the **generalized Lorenz curve** for X:

$$GL_X(p) := \mu(X) \cdot L_X(p).$$

Income profile X generalize *Lorenz dominates* income profile Y, $X \succcurlyeq_{GL} Y$, if and only if

$$GL_X(p) \geq GL_Y(p) \text{ for all } p \in [0,1].$$

Kolm (1969), Shorrocks (1983).

- If $\mu(X) = \mu(Y)$, $\succcurlyeq_{GL} \iff \succcurlyeq_L$;
- $X/\mu_X \succcurlyeq_{GL} Y/\mu_Y \iff X \succcurlyeq_L Y$.

$$GL_x(i/n) = \frac{1}{n}\sum_{j=1}^{i} \hat{x}_{(j)} \qquad \text{where } \hat{x}_{(i)} \leq \hat{x}_{(i+1)}$$

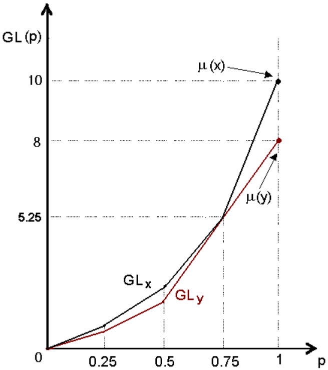


Figure: Generalized Lorenz Curves for $\mathbf{y} = (3, 7, 11, 11)$, $\mathbf{x} = (4, 8, 9, 19)$

no-dominance if the GL curves intersect

Relation with more general results on unidimensional inequality and welfare

- $I(\mathbf{x})$ is continuous and normalized $I(\mu, \mu, .., \mu) = 0$.
- [Symmetry (S)] $I(\mathbf{x})$ is invariant with respect to permutation of the incomes.
- [Pigou-Dalton Principle of Transfers (PT)] A transfer from a rich person (j) to a poor person (i) which leaves their relative positions unchanged reduces inequality

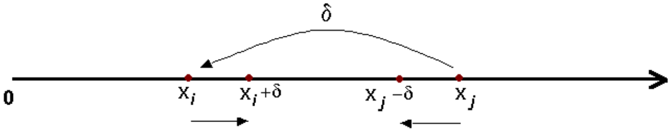


Figure: Progressive Transfer

- [Relative Inequality (Rel)] $I(\mathbf{x}) = I(\lambda \mathbf{x})$ for $\lambda > 0$.

Theorem (Hardy, Littlewood & Polya 1934 (HL&P))

Consider a fixed number of individuals n, let $\mu(\mathbf{x}) = \mu(\mathbf{y})$, *the following statements are equivalent:*

(1) For all $k \leq n$, $\sum_{i=1}^{k} \hat{x}_i \geq \sum_{i=1}^{k} \hat{y}_i$ *with at least one strict inequality* $(>)$.

(2) $\hat{\mathbf{x}}$ *can be obtained from* $\hat{\mathbf{y}}$ *through a finite sequence of progressive transfers.*

(3) Let $W_u(\mathbf{x}) = \sum_{i=1}^{n} u(x_i)$ *the Utilitarian Social Evaluation Function,* $W_u(\mathbf{x}) > W_u(\mathbf{y})$ *for all* $W_u(\mathbf{x})$ *such that* $u(.)$ *is increasing and strictly concave.*

(4) Let $I_\phi(\mathbf{x}) = \sum_{i=1}^{n} \phi(x_i)$ *the additive inequality index* $I_\phi(\mathbf{x}) < I_\phi(\mathbf{y})$ *for all* $I_\phi(\mathbf{x})$ *such that* $\phi(.)$ *is strictly convex.*

Theorem

(5) $\mathbf{x} = \Pi\mathbf{y}$ *where* Π *is a* $n \times n$ *bistochastic matrix.*

Links between Inequality & Welfare

We consider Social Evaluation Function (SEF) $W(\mathbf{x}) : \mathbb{R}^n_+ \rightarrow \mathbb{R}$

- [Inequality - Welfare Consistency (IWC)] If $\mu(\mathbf{x}) = \mu(\mathbf{y})$ then for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$

$$I(\mathbf{x}) \leq I(\mathbf{y}) \Leftrightarrow W(\mathbf{x}) \geq W(\mathbf{y}).$$

Theorem (Shorrocks (1983); Kolm (1969))

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$ the following statements are equivalent:
(i) $\mathbf{x} \succ_{GL} \mathbf{y}$
(ii) $W(\mathbf{x}) > W(\mathbf{y})$ for all increasing SEFs $W(\mathbf{x})$ satisfying Symmetry,and Principle of Transfers.
(iii) $\frac{1}{n}\sum_{i=1}^{n} u(x_i) > \frac{1}{n}\sum_{i=1}^{n} u(y_i)$ for all Average Utilitarian SEFs where $u(.)$ is increasing and strictly concave.

WE NOW MOVE TO THE MULTIDIMENSIONAL CASE...

Distribution $X \in \mathbb{R}_+^{n\times d}$

$$X = \begin{bmatrix} x_{11} & \cdots & \cdots & x_{1(d-1)} & x_{1d} \\ x_{21} & x_{22} & \cdots & \cdots & x_{2d} \\ \vdots & \cdot & x_{ij} & & \vdots \\ x_{n1} & x_{n2} & \cdots & \cdots & x_{nd} \end{bmatrix}$$

- $\mathbf{x}_{.j} \in \mathbb{R}_+^n$ distribution of attribute j across all individuals, (column)
- $\mathbf{x}_{i.} \in \mathbb{R}_+^d$ distribution of all the attributes for individual i, (row)

Cross iterative aggregating procedures

Consistency in Aggregation

Rubinstein, Fishburn (JET 1986): Algebraic aggregation theory

Dutta, Pattanaik, Xu (Economica 2003): On Measuring Deprivation and the Standard of Living in a Multidimensional Framework on the Basis of Aggregate Data.

Gajdos, Maurin (JET 2004): Unequal uncertainties of uncertain inequalities: an axiomatic approach

Question: Is it possible to obtain consistent ranking across matrices aggregating......

1. first for each attribute taking into account the distribution across agents and then aggregating the summary Macro result across attributes (Procedure 1) (e.g *HDI*)
2. for each agent obtaining an individual index of personal well being and then aggregating the distribution of these indices for all the population (Procedure 2) (e.g.additively decomposable SWFs over multiattribute distributions)......

Consistent iterative aggregation

.... moreover one would like to apply the same aggregator in each procedure when aggregating across individuals and another on when aggregating across attributes.$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$

Procedure 1:

$$\begin{matrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{matrix} \Downarrow \\ Aggr.\ \psi \end{matrix} \\ \downarrow \\ \begin{bmatrix} \psi(x_{11}; x_{21}) & \psi(x_{12}; x_{22}) \end{bmatrix} \\ \Longrightarrow Aggregator\ \phi \end{matrix} \longmapsto \phi[\psi(x_{11}; x_{21}); \psi(x_{12}; x_{22})]$$

Procedure 2:

$$\overset{\Longrightarrow \phi}{\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}} \longmapsto \begin{bmatrix} \phi(x_{11}; x_{12}) \\ \phi(x_{21}; x_{22}) \end{bmatrix} \begin{matrix} \Downarrow \\ \psi \end{matrix} \longmapsto \psi[\phi(x_{11}; x_{12}); \phi(x_{21}; x_{22})]$$

Consistent aggregation

A result..Dutta et al. (2003): Assumptions

Definition (Consistency)

$\phi \circ \psi(X) \geq \phi \circ \psi(Y)$ iff $\psi \circ \phi(X) \geq \psi \circ \phi(Y)$

moreover suppose that $x_{ij} \in [c_{\min}; c^{\max}]$ and
$\phi : [c_{\min}; c^{\max}]^d \rightarrow [0,1]$; $\phi(\mathbf{1}c_{\min}) = 0$; $\phi(\mathbf{1}c^{\max}) = 1$
$\psi : [c_{\min}; c^{\max}]^n \rightarrow [0,1] : \psi(\mathbf{1}c_{\min}) = 0$; $\psi(\mathbf{1}c^{\max}) = 1$
ϕ and ψ are continuous and strictly increasing in each argument
ψ is symmetric across agents
ϕ exhibit **non increasing increments**

$$\begin{aligned} & \phi(x_1, x_2, .., x_h, ..., x_k + t, ..x_d) - \phi(x_1, x_2, .., x_h..., x_k, ..x_d) \\ \leq \;& \phi(x_1, x_2, .., x_h - \tau, ..., x_k + t, ..x_d) - \phi(x_1, x_2, .., x_h - \tau, ..., x_k, ..x_d) \end{aligned}$$

for $\tau, t > 0$

Consistent aggregation

A result..Dutta et al. (2003):

Theorem

Given the assumptions on ϕ and ψ, the two procedures are consistent iff

$$\phi(x_{i.}) = \frac{\sum_{j=1}^{d} w_j \cdot x_{ij} - c_{\min}}{c^{\max} - c_{\min}}; \text{ where } w_j > 0, \sum_{j=1}^{d} w_j = 1$$

$$\psi(x_{.j}) = \frac{\frac{1}{n}\sum_{i=1}^{n} x_{ij} - c_{\min}}{c^{\max} - c_{\min}}$$

We obtain essentially HDI types of indices.

Correlation between attributes is lost $Y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $Z = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ are socially indifferent

Consistent aggregation.... Note that...

The assumption of non increasing increments (NII) per each agent across attributes is crucial. For instance

Example

Check that for given increasing functions $f_j : [c_{\min}; c^{\max}] \longrightarrow [0,1]$: $f(c_{\min}) = 0$; $f(c^{\max}) = 1$ and the increasing function $H : [0,1] \longrightarrow [0,1]$ the following functional forms satisfy consistency and all other properties but not necessarily NII

$$\phi(x_{i.}) = H^{-1}(\sum_{j=1}^{d} w_j \cdot H[f_j(x_{ij})]); \text{ where } w_j > 0, \ \sum_{j=1}^{d} w_j = 1$$

$$\psi(x_{.j}) = H^{-1}(\frac{1}{n}\sum_{i=1}^{n} H[f_j(x_{ij})])$$

Property NII imposes linearity in H and f_j.

Consistent aggregation

An example without NII..

Foster et. al. (JED 2005) consider the case where H is isoelastic $H(t) = t^{1-\varepsilon}/(1-\varepsilon)$ for $\varepsilon \geq 0$ and $w_j = 1/d$.
if we let $s_{ij} = f_j(x_{ij})$ the index "consistent in aggregation" is obtained for

$$\phi(x_{i.}) = \left(\frac{1}{d}\sum_{j=1}^{d}[s_{ij}]^{1-\varepsilon}\right)^{1/(1-\varepsilon)} ; \ \psi(x_{.j}) = \left(\frac{1}{n}\sum_{i=1}^{n}[s_{ij}]^{1-\varepsilon}\right)^{1/(1-\varepsilon)}$$

$$I(X) = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{d}\sum_{j=1}^{d}[s_{ij}]^{1-\varepsilon}\right)^{1/(1-\varepsilon)} .$$

then H satisfies NII in terms of the distribution of y iff $\varepsilon = 0$.
Main positive features of the index is that it is Subgroup Consistent $I(X,Y) \geq I(X,Z)$ iff $I(Y) \geq I(Z)$. Where (X,Y) denotes that the population is partitioned into two groups of individuals.

May be one can take an average of the results arising from the two procedures:

This has already been suggested in the literature on the measurement of inequality under uncertainty

Gilboa and Schmeidler (JMathE 1989), Ben Porath et al. (JET 1997) Gajdos and Maurin (JET 2004) Gajdos and Weymark (ET 2005)

suppose for simplicity that we normalize each agent realization in a given space with $s_{ij} := f(x_{ij}) \in [0, 1]$: score function associated with the realization of agent i on space j. Moreover

- this normalization makes comparable scores of the same agent in different characteristics (Symmetry between characteristics)
- and agents are all treated equally in the final evaluation (Anonymity)

Alternative sequences.. Examples

Consider the following distributions:

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}; X' = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix};$$

if attributes carry the same weight $W(X) = W(X')$

$$Z = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}; Z' = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix};$$

if agents are equally relevant $W(Z) = W(Z')$

$$Y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; Y' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$$

if both previous considerations hold then $W(Y) = W(Y')$

However linear symmetric aggregation value as indifferent all distributions.

But this should not be the case in particular comparing Y, Y' with all others

Alternative sequences..

problems with consistent additive decomposition

The previous considerations extend also to the additive decomposition of the matrices as in Foster et al. (2005) or even in the more general result presented earlier on consistent aggregation.... The procedure was including considerations on inequality in the distribution across agents but **what is left aside is the correlation between the distributions of the attributes**

$$Z = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} ; Y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} ; X = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

- It is possible to regain some considerations if we give up the issue of consistency in aggregation...specifying a given order and in an additive framework consider different parameters ε in aggregating across distribution w.r.t. those applied in aggregating across individuals. (Decanq, Decoster Schokkaert World Dev 2008)

The general issue is the increase in correlation between attributes

From $Y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ to $Z = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ we have transferred attribute 1 from agent 2 to agent 1 that now clearly dominates the latter.

Definition (CIT)

In general a **Correlation Increasing Transfer** $CIT\ (i, j)$ is a sequence of "rearrangements" of the distribution of attributes (one attribute per step of the sequence) involving only two individuals (i, j) s.t. as the result of the process one individual ends up being weakly dominated by the other in any attribute.

Epstein and Tanny (CanJEc1980), Tsui (JET 1995, SCW 1999)

Example

A sequence of CIT

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \underset{CIT(13)}{\Longrightarrow} \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 3 & 3 \end{bmatrix} \underset{CIT(12)}{\Longrightarrow} \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 2 \\ 2 & 3 & 3 \end{bmatrix} \underset{CIT(23)}{\Longrightarrow} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$$

The last matrix (Z') is more unfair than the first one (Y') looking from the individuals perspective

All indices first aggregating across attributes will consider $W(Y') = W(Z')$

Also inequality indices aggregating first across individuals may lead to $W(Y') < W(Z')$

See Dardanoni (1995) comment on Maasaouni (1986) Multidimensional Inequality Index

Alternative sequences.. Using non additive measure (Gini type welfare/inequality indices)

Use a functional $\psi = G$ in order to aggregate across individuals their realizations in each attribute [aggregating vertically for each matrix column] and another functional $\phi = \mu$ in order to aggregate across attributes per each individual [aggregating horizontally for each matrix row].
Then evaluate $\mu \circ G$ and $G \circ \mu$ **and take their weighted average.**

Definition

First procedure $\mu \circ G$ (average of Gini indices of the agents distribution for each attribute): aggregating for each attribute taking the Gini index across agents AND then averaging the obtained result.
Second procedure (Gini index of average agent score).

Comments

Note the procedures are different from Hicks (1997 World Dev.) proposal of taking the average of the Gini welfare index of the distribution of each attribute.

Gajdos and Weymark (ET 2005) characterize families of Generalized averages across attributes of Gini indices across individuals per each attribute.

However as already pointed out for these measures

$$Z = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

are considered socially indifferent.

What about applying a Gini evaluation to the overall matrix?

A problem due to comonotonic independence (across attributes):

$$H = \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix}; H' = \begin{bmatrix} 0 & 0 \\ 1/2 & 0 \end{bmatrix}; K = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 0 \end{bmatrix};$$

K is comonotonic w.r.t. H and H' because the ranking of the attributes is the same in all matrices
Taken any pair of cells ij and $i'j'$ it is always true that
$H_{ij} \geq H_{i'j'} \iff K_{ij} \geq K_{i'j'}$ (similarly for comparisons of K and H')

- Comonotonic Independence is the key property characterizing Gini type (i.e. rank dependent) evaluations!

by Comonotonic independence between K and H and K and H'.
It follows that for a Gini index G

$$G(H+K) \geq G(H'+K) \Longleftrightarrow G(H) \geq G(H')$$

...but $G(H) = G(H')$ by anonymity, thus

$$\begin{aligned} G(H+K) &= G(H'+K) \quad \textit{where} \\ H+K &= \begin{bmatrix} 1 & 1/2 \\ 1/2 & 0 \end{bmatrix}; H'+K = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}. \end{aligned}$$

However "according to CIT" we need to have
$W(H+K) \leq W(H'+K)$.

- Open question...appropriate definition of multidimensional Gini functionals...

From indices to dominance conditions

The key component is still the dependence between attributes
Different tools:

- Multidimensional Majorization
- Multidimensional version of Lorenz and Generalized Lorenz curves
- Multidimensional Stochastic dominance conditions

Multidimensional Majorization

Kolm (QJE 1977), Koshevoy (SCW 1995), Koshevoy, Mosler (JASA 1996), Koshevoy, Mosler (AStA 2007), Weymark (2004), Savaglio (2004)

Marshall, A. W. and Olkin, I. (1979): Inequalities: Theory of Majorization and Its Applications. New York: Academic Press.

Generalizes the majorization from the unidimensional case to multidimensional distributions where attributes are transferable between individuals (e.g. bundles of goods) Marshall and Olkin (1979)

Multidimensional Majorization

Consider matrices $X, Y \in \mathbb{R}^{n \times d}$ for n individuals, d goods (characteristics)
and such that $\sum_{i=1}^{n} \mathbf{x}_i = \sum_{i=1}^{n} \mathbf{y}_i$ where $\mathbf{x}_i, \mathbf{y}_i \in \mathbb{R}^d$ vectors of goods belonging to individual i.

- Distribution of a fixed bundle of d goods across n individuals

Definitions

Y multidimensionally majorize X,

$$Y >_M X \Longleftrightarrow X = \Pi Y,$$

where Π is a $n \times n$ bistochastic matrix.

X is obtained from Y averaging the endowments vectors of the individuals, or in other terms X shows less disparity in the distributions of the bundles of goods than Y.

Multidimensional Majorization

Example

$$\Pi = \begin{bmatrix} 0.1 & 0.2 & 0.7 \\ 0.2 & 0.6 & 0.2 \\ 0.7 & 0.2 & 0.1 \end{bmatrix}; Y = \begin{bmatrix} 10 & 30 \\ 20 & 30 \\ 10 & 0 \end{bmatrix} \Longrightarrow X = \Pi Y = \begin{bmatrix} 12 & 9 \\ 16 & 24 \\ 12 & 27 \end{bmatrix}$$

individual 3 situation in X is substantially improved (she has overtaken individual 1) if compared to Y.
but we need to rely on symmetric evaluations across individuals (a permutation matrix is bistochastic)
Note however that all attributes are "mixed" in the same way for each individual (i.e. they are multiplied by the same raw of Π)
$\mathbf{x}_i = \sum_{k=1}^{n} \pi_{ik} \cdot \mathbf{y}_k$ the averaging of every attribute is made using the same weights depending on the individuals and not on the attribute itself.

Multidimensional Majorization

A welfare interpretation

Consider matrices $X, Y \in \mathbb{R}^{n \times d}$ such that $\sum_{i=1}^{n} \mathbf{x}_i = \sum_{i=1}^{n} \mathbf{y}_i$

Theorem

The following conditions are equivalent:
(I) $Y >_M X$
(II) $\phi(Y) \leq \phi(X)$ *for all* $\phi : \mathbb{R}^{n \times d} \rightarrow \mathbb{R}$ *which are S-concave;*
(III) $\sum_{i=1}^{n} u(\mathbf{y}_{i.}) \leq \sum_{i=1}^{n} u(\mathbf{x}_{i.})$ *for all* $u : \mathbb{R}^{d} \rightarrow \mathbb{R}$ *which are concave [they can also be increasing].*

S-Concave function: symmetric functions such that $\phi(Y) \leq \phi(\Pi Y)$
The distribution X of a fixed amount of resources improves welfare

- Does it means that we have also less inequality in terms of the distribution of the concave and increasing utilities $u(\mathbf{y}_i)$?
- Will it be possible to decompose the change from Y to X in terms of progressive transfers?

Multidimensional Majorization

A controversial implication in terms of inequality

Dardanoni (REI 1994) On multidimensional inequality measurement

Example

$$\Pi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}; Y = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 3 & 1 \end{bmatrix} \Longrightarrow X = \Pi Y = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 2 & 2 \end{bmatrix}$$

The utility of the poorest individual 1 is left unchanged, while according to any strictly concave utility agents 2 and 3 are better off. There is more inequality as well as welfare even though the resources are fixed!!!

- If we consider majorization dominance as ethical compelling then in evaluating inequality the approach that first aggregates in terms of individual utilities might not be appropriate

Multidimensional Majorization

Relation with Progressive (Pigou Dalton) transfers

Will it be possible to decompose the change from Y to X in terms of progressive transfers?

Definition

T transform (Pigou Dalton transfer)

$$\begin{aligned}\Pi_{2,3}(\lambda) &= \lambda \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (1-\lambda) \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & (1-\lambda) \\ 0 & (1-\lambda) & \lambda \end{bmatrix}\end{aligned}$$

a convex combination of identity matrix and a permutation matrix involving a permutation of 2 individuals.

Multidimensional Majorization

Relation with Progressive (Pigou Dalton) transfers

Will it be possible to decompose the change from Y to X in terms of progressive transfers?

In general when $n \geq 3$ not all bistochastic matrices Π can be obtained as product of T transforms

this issue is particularly crucial when $d \geq 2$.

Example

the matrix $\Pi = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix}$, cannot be obtained as product of T-transforms

the 3 entries with 0 involving all individuals cannot be replicated by a chain of T-transforms different from permutation matrices in which case we won't be able to obtain the 0.5 entries

Multidimensional Majorization

Relation with Progressive (Pigou Dalton) transfers

Note however that in the unidimensional case the above mentioned problem is not an issue because there exist always the possibility to obtain the final distribution through T transforms even though they generate a different bistochastic matrix:

Example

$$\begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 30 \\ 30 \\ 0 \end{bmatrix} = \begin{bmatrix} 15 \\ 15 \\ 30 \end{bmatrix} = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 30 \\ 30 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 30 \\ 30 \\ 0 \end{bmatrix}$$

With two dimension it is essential to be able to generate the specific bistochastic matrix considered.

Tests for Multidimensional Majorization...

- Unfortunately SO FAR there exists no multidimensional generalization of the Lorenz curve which could be used to rank matrices according to the $>_M$ preorder.

Standard majorization could be wakened in different ways in order to be applied in different economic contexts. The most interesting device is the *price majorization* criterion Kolm (QJE 1977).

Definitions

The matrix Y is said to price majorize X that is

$$Y >_P X \iff Yp >_M Xp \; \forall p \in \mathbb{R}^d_+, \; (or \; \forall p \in \mathbb{R}^d)$$

i.e. the distribution of potential incomes associated to X, and evaluated according to the vector of prices p, Lorenz dominates the one associated to Y for all possible price profiles [they can be also negative].

Price Majorization...

According to $>_P$ the distribution of initial endowments X is always preferred to Y by an inequality averse policy maker which is concerned in maximizing the distribution of indirect utilities and attaches to each individual the same direct utility function, this evaluation is valid no matter what will be the equilibrium price profile.

$$Y >_M X \Leftrightarrow X = \Pi Y \Longrightarrow Xp = \Pi Yp \Leftrightarrow Yp >_M Xp \Leftrightarrow Y >_P X$$

- thus $>_M \Longrightarrow >_P$ but the converse is not always true.

Fact

There exist multidimensional generalizations of the Lorenz curve that can be used to test $>_P$ both when $p \in \mathbb{R}^d_+$, and $p \in \mathbb{R}^d$. They also work as analogous of generalized Lorenz dominance over distributions of income budgets. The are the Lorenz Zonoid, the Lift Zonoid and their extensions!!!

Note that...

Price majorization can be useful to rank $Z = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Consider $p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$, if either $p \in R^2_{++}$ or $p \in R^2_{--}$ we have

$$Yp = \begin{bmatrix} p_2 \\ p_1 \end{bmatrix} = \begin{bmatrix} \lambda & (1-\lambda) \\ (1-\lambda) & \lambda \end{bmatrix} \begin{bmatrix} p_1 + p_2 \\ 0 \end{bmatrix} = Zp$$

$$\text{when } \lambda = \frac{p_2}{p_1 + p_2} \in (0, 1)$$

thus $Zp >_M Yp$.

- *Price majorization with positive prices* appears an interesting candidate for a meaningful multidimensional dominance condition.

Lift Zonoid and Lorenz Zonoid

by Koshevoy & Mosler

For empirical distributions:

Definitions

The Lift Zonoid $Z(X)$ is a convex compact set in the $(d+1)$ space obtained as the weighted sum of segments $\mathbf{x}_i \in \mathbb{R}^d$, for all possible sets of normalized weights, that is

$$Z(X) = \left\{ \sum_{i=1}^{n} z_{0i}; \sum_{i=1}^{n} z_{0i}\mathbf{x}_i : 0 \leq z_{0i} \leq 1/n, \; i = 1, 2, ..., n \right\}.$$

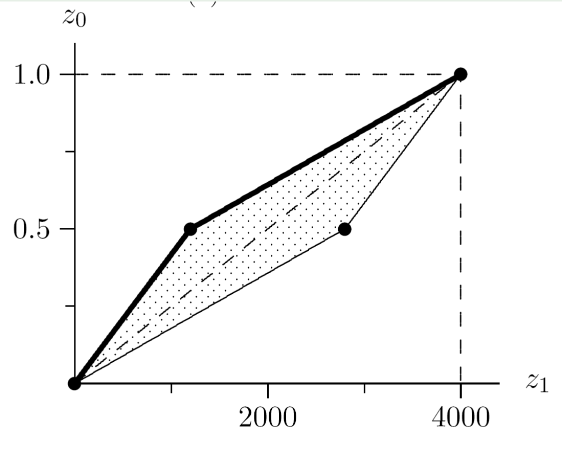
The Lorenz Zonoid $LZ(X)$ is the Lift Zonoid evaluated over distribution $\tilde{X} := (\mathbf{x}_{.1}/\mu(\mathbf{x}_{.1}); \mathbf{x}_{.2}/\mu(\mathbf{x}_{.2}); ...; \mathbf{x}_{.d}/\mu(\mathbf{x}_{.d}))$ where each attribute is normalized dividing it by its average: $LZ(X) := Z(\tilde{X})$.

Zonoids the intuition

- Take all subsets of the population of a given relative size $z_0 \in [0,1]$ (e.g. 50%) [where $z_0 = \Sigma_i z_{0i}$] what is the aggregate (divided by n) realization in the d dimensional space of the resources of any of these subsets covering a z_0 proportion of population? Take the convex hull of all these distributions in the d dimensional space. We have obtained the **section of the Lift Zonoid for a fixed value of population share** z_0. As z_0 moves from 0 to 1 we construct the Lift Zonoid. For finite populations we convexify all sections corresponding to adjacent proportions of population.
- In order to get the Lorenz Zonoid we need just to apply the same logic to the normalized distributions of each attribute. So normalize each columns in relative terms so that any aggregate amount sums to n.

Example

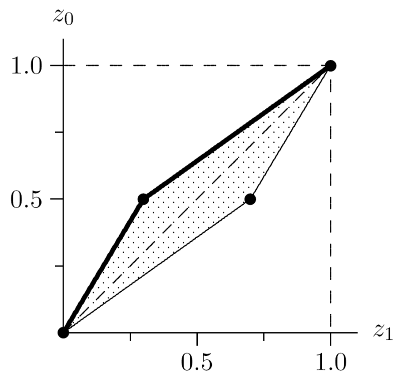
Lift zonoid of univariate distribution (2400,5600) taken from Koshevoy and Mosler (AStA2007)



(b) Lift zonoid

Example

Lorenz zonoid of univariate distribution (2400,5600) [in relative terms (0.6; 1.4)] taken from Koshevoy and Mosler (AStA2007)



(a) Lorenz zonoid

Dominance is defined in terms of inclusion of Zonoids

Example

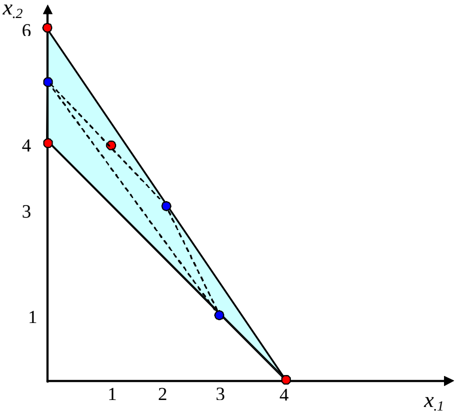
$$Y = \begin{bmatrix} 0 & 6 \\ 0 & 4 \\ 4 & 0 \\ 1 & 4 \end{bmatrix} \quad X = \begin{bmatrix} 0 & 5 \\ 0 & 5 \\ 2 & 3 \\ 3 & 1 \end{bmatrix} \text{ note that } Z(X) \subseteq Z(Y).$$

Note that we have (Generalized) Lorenz dominance for each attribute!!

There is no (4×4) bistochastic matrix Π such that $X = \Pi Y$
In order to accommodate for the transformation from Y to X involving the first two individuals the only admissible matrix should be $\begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}$. If we set $a = 1/3; b = 2/3$ in order to accommodate for the first attribute of the third individual we cannot obtain the distribution in X of her second attribute!!!

A graphical representation for $Z(X) \subseteq Z(Y)$ evaluated for $z_0 = 1/4$

$$Y = \begin{bmatrix} 0 & 6 \\ 0 & 4 \\ 4 & 0 \\ 1 & 4 \end{bmatrix} \quad X = \begin{bmatrix} 0 & 5 \\ 0 & 5 \\ 2 & 3 \\ 3 & 1 \end{bmatrix}$$

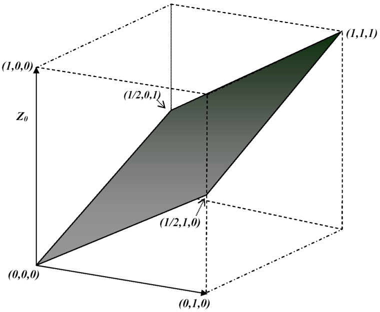


A further example

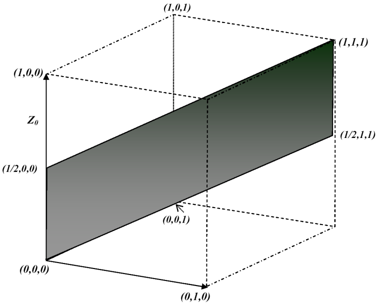
Zonoids fail in taking into account the effect of Correlation Incres. Transf.

$$Z = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}; \; Y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

There is no inclusion relation between the Lift Zonoids of the two distributions $Z(Z)$ and $Z(Y)$.



Lift Zonoid of Y



Lift Zonoid of Z

Lorenz Dominance and Price Majorization

Theorem

The following conditions are equivalent:
(I) $LZ(X) \subseteq LZ(Y)$.
(Ia) $LZ(X_S) \subseteq LZ(Y_S)$ *for the distributions of all the subsets S of attributes*
(II) $\tilde{Y} >_P \tilde{X}$ *(for all* $p \in \mathbb{R}^d$*).*
(III) $\tilde{X}p$ *Generalize Lorenz dominates* $\tilde{Y}p$ *for all* $p \in \mathbb{R}^d$*. (Budget dominance)*
(IV) $\psi(\tilde{Y}p) \leq \phi(\tilde{X}p)$ *for all* $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ *which are S-concave and all* $p \in \mathbb{R}^d$*.*
(V) $\sum_{i=1}^n v(\tilde{\mathbf{y}}_i \cdot p) \leq \sum_{i=1}^n v(\tilde{\mathbf{x}}_i \cdot p)$ *for all* $v : \mathbb{R} \rightarrow \mathbb{R}$ *which are concave and all* $p \in \mathbb{R}^d$*.*

- Is it possible to obtain conditions analogous to Generalized Lorenz dominance of budgets when prices are only positive?

We need to extend the notion of Lift Zonoid.

Definition

The **Extended Lift Zonoid** $eZ(X)$ is obtained extending the volume of the Lift Zonoid taking all points below it for the coordinate relative to the population share and all points above the Lift Zonoid in the d dimensional space of the attributes. Thus for instance any two dimensional section of $Z(X)$ for a given population share taking all points north-east w.r.t. each point in $Z(X)$. In general $eZ(X) := Z(X) + (\mathbb{R}_{-} \times \mathbb{R}_{+}^{d})$.

Price dominance with positive prices can be implemented through the Extended Lift Zonoid also for distributions with different total amounts of attributes......

Budget dominance with positive prices

The result

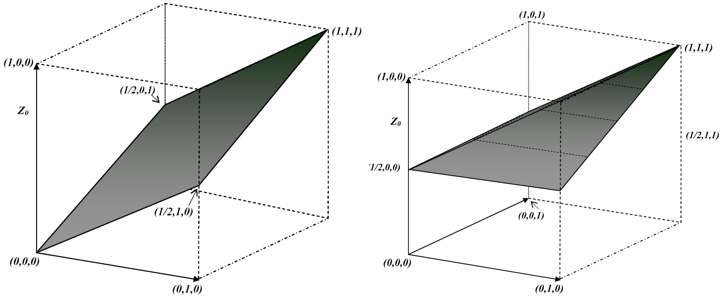
Theorem

The following conditions are equivalent:
(I) $eZ(X) \subseteq eZ(Y)$.
(II) Xp *Generalize Lorenz dominates* Yp *for all* $p \in \mathbb{R}^d_+$. *(Budget dominance)*
(III) $\psi(Yp) \leq \phi(Xp)$ *for all* $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ *which are increasing and S-concave and all* $p \in \mathbb{R}^d_+$.
(IV) $\sum_{i=1}^n v(\mathbf{y}_i \cdot p) \leq \sum_{i=1}^n v(\mathbf{x}_i \cdot p)$ *for all* $v : \mathbb{R}_+ \rightarrow \mathbb{R}$ *which are increasing and concave and all* $p \in \mathbb{R}^d_+$.

If for each share of population the upper contour set of the section of the Lift Zonoid of X in the d dimensional space is included into the same set for Y then social welfare is larger in X than in Y for SWFs that are increasing and inequality averse.

Back to the example

For $Z = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ we know that $Zp >_M Yp$ for $p \in \mathbb{R}^2_+$ thus according to the previous theorem it is also true that $eZ(Y) \subseteq eZ(Z)$. One can check this last condition with the graphical representation.



$eLZ(Y)$: volume below the $eLZ(Z)$: volume below the

Multidimensional Stochastic orders

Integral Stochastic Orders can be applied in a multidimensional framework for distributions in $\mathbb{R}_+^d$.
We focus on very few stochastic orders see Shaked and Shanthikumar (1994) and Müller and Stoyan (2002) for a survey.
Let $X = (X_{.j} : j = 1, 2, ..d)$ denote the marginal distributions of each attribute j with generic realization $\mathbf{x} = (x_1, x_2, ..., x_d)$ identifying a d dimensional vector of realizations one for each attribute with

- Cumulative Distribution Function:

$$F_{\mathbf{X}}(\mathbf{x}) := P(X \leq \mathbf{x}) := P(X_{.j} \leq x_j \text{ for all } j = 1, 2, 3, ..., d)$$

  with marginals $F_{X_{.j}}(x)$ for $j = 1, 2, 3, ..., d$.
- Survival Function:

$$\bar{F}_{\mathbf{X}}(\mathbf{x}) := P(X > \mathbf{x}) := P(X_{.j} > x_j \text{ for all } j = 1, 2, 3, ..., d)$$

- Here probabilities logically correspond to proportions of populations in the multidimensional distribution framework.

Multidimensional Stochastic orders

A first result

- Let $u : \mathbb{R}_+^d \to \mathbb{R}$ denote an "utility" [evaluation] function over d dimensional attributes realizations $\mathbf{x}$ and

$$^d\mathcal{U}_1 := \{u : u \text{ non-decreasing}\}$$

i.e. if $\mathbf{x} \geq \mathbf{x}'$ then $u(\mathbf{x}) \geq u(\mathbf{x}')$. Thus $\geqslant_{^d\mathcal{U}_1}$ denotes the following integral stochastic order

Definition

$$X \geqslant_{^d\mathcal{U}_1} Y \Longleftrightarrow \int_{\mathbb{R}_+^d} udF_{\mathbf{X}} \geq \int_{\mathbb{R}_+^d} udF_{\mathbf{Y}} \;\; \forall u \in {}^d\mathcal{U}_1$$

Upper sets and dominance

Definition (Upper Set)

The set $U \in \mathbb{R}^d_+$ is an upper set iff for all $\mathbf{x} \in \mathbb{R}^d_+$ if $\mathbf{x} \in U$ then $\mathbf{y} \in U$ if $\mathbf{y} \geq \mathbf{x}$.

Theorem

The following statements are equivalent:
(i) $X \geqslant_{d\mathcal{U}_1} Y$
(ii) $P(X \in U) \geq P(Y \in U)$ *for all upper set* U *in* $\mathbb{R}^d_+$.

Definitions

let $\mathbb{P} := \{p \in \mathbb{R}^d_+ : p_1 + p_2 + ... + p_d = 1\}$

$$X \geqslant^1_{\mathbb{P}} Y \iff \int_{\mathbb{R}^d_+} g(Xp)dF_{\mathbf{X}} \geq \int_{\mathbb{R}^d_+} g(Yp)dF_{\mathbf{Y}} \ \ \forall g \in {}^1\mathcal{U}_1 \ \forall p \in \Delta$$

The following statements are equivalent (Muliere and Scarsini EcLett 1989):

Theorem

(i) $X \geqslant^1_{\mathbb{P}} Y$
(ii) $P(Xp > t) \geq P((Yp > t)$ *for all* $t > 0$. *(Dominance for all upper sets whose boundary is an Hyperplane)*

Problem

Is it possible to add to the list of equivalent conditions also those on comparisons of $F_{\mathbf{X}}$ and $F_{\mathbf{Y}}$ or of $\bar{F}_{\mathbf{X}}$ and $\bar{F}_{\mathbf{Y}}$?

Definition (Upper Orthant order)

$X \succcurlyeq_{uo} Y \iff \bar{F}_{\mathbf{X}}(\mathbf{t}) \geq \bar{F}_{\mathbf{Y}}(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{R}_+^d$.

Definition (Lower Orthant order)

$X \succcurlyeq_{lo} Y \iff F_{\mathbf{X}}(\mathbf{t}) \leq F_{\mathbf{Y}}(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{R}_+^d$.

- $X \succcurlyeq_{uo} Y$ and $X \succcurlyeq_{lo} Y$ are independent.
- $X \geqslant_{d\mathcal{U}_1} Y \Longrightarrow [X \succcurlyeq_{uo} Y$ and $X \succcurlyeq_{lo} Y]$
- $[X \succcurlyeq_{uo} Y$ and $X \succcurlyeq_{lo} Y] \not\Longrightarrow X \geqslant_{d\mathcal{U}_1} Y$.
- $[X \succcurlyeq_{uo} Y$ and $X \succcurlyeq_{lo} Y]$ give the Concordance order $X \succcurlyeq_c Y$, i.e. an order of association between variables

From last remark clearly the answer to the previous problem is NO!

Note that by construction Upper Sets are unions of Upper Orthants.

Definition ($\Delta-$monotone functions)

Consider the function $u : \mathbb{R}_+^d \rightarrow \mathbb{R}$, let $\varepsilon > 0$, $\mathbf{1} := (1, 1, 1, 1, ..., 1)$ and $\mathbf{1}_i := (0, 0, 0, 1_i, 0, ..., 0)$

$$\Delta_i^{\varepsilon} u (\mathbf{x}) := u (\mathbf{x} + \varepsilon \mathbf{1}_i) - u (\mathbf{x}) .$$

Function u is $\Delta-$monotone if for every set $\{i_1, i_2, .., i_k\} \subset \{1, 2, 3, ..., d\}$ and every $\varepsilon_i > 0$ for $i \in \{1, 2, 3, .., k\}$ then

$$\Delta_{i_1}^{\varepsilon_1} \Delta_{i_2}^{\varepsilon_2} ... \Delta_{i_k}^{\varepsilon_k} u (\mathbf{x}) \geq 0.$$

Delta monotone functions

Definition

${}^{d}\Delta_{\mathcal{M}}$ is the set of all bounded $\Delta-$monotone functions $u : \mathbb{R}^{d}_{+} \to \mathbb{R}$.

Definition

${}^{d}\Delta_{\mathcal{A}}$ is the set of all bounded $\Delta -$ *antitone* functions $u : \mathbb{R}^{d}_{+} \to \mathbb{R}$ i.e. $u(\mathbf{x}) \in {}^{d}\Delta_{\mathcal{A}} \Leftrightarrow -u(-\mathbf{x}) \in {}^{d}\Delta_{\mathcal{M}}$. NB $\Delta -$ *antitone* functions satisfy **decreasing increments.**

Theorem

(i) $X \succcurlyeq_{uo} Y \iff X \geqslant_{{}^{d}\Delta_{\mathcal{M}}} Y$
(ii) $X \succcurlyeq_{lo} Y \iff X \geqslant_{{}^{d}\Delta_{\mathcal{A}}} Y$.

Definition (Supermodular functions)

The function $u : \mathbb{R}_+^d \to \mathbb{R}$, is supermodular if for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^d$

$$\begin{aligned} & u\left(\max\{x_1, y_1\}; ..; \max\{x_d, y_d\}\right) + u\left(\min\{x_1, y_1\}; ..; \min\{x_d, y_d\}\right) \\ \geq \; & u\left(\mathbf{x}\right) + u\left(\mathbf{y}\right). \end{aligned}$$

alternatively if u is twice differentiable $\frac{\partial^2 u}{\partial x_i \partial x_j} \geq 0$ for all $i, j \in \{1, 2, 3, .., d\}$, $i \neq j$.

Fact

A function $u : \mathbb{R}_+^d \to \mathbb{R}$ is supermodular if and only if $\int_{\mathbb{R}_+^d} u dF_{\mathbf{X}} \geq \int_{\mathbb{R}_+^d} u dF_{\mathbf{Y}}$ whenever X is obtained from Y through a Correlation Increasing Transformation. (It is an indicator of dependence across attributes)

Supermodular functions

Definition

The set of supermodular functions is ${}^{d}\mathcal{U}_{SM}$.

Consider the bivariate case:

Theorem

Let $d = 2$, the following statements are equivalent:
(i) $X \geqslant_{{}^{2}\mathcal{U}_{SM} \cap {}^{2}\mathcal{U}_{1}} Y$
(ii) $X \succcurlyeq_{uo} Y$ *and* $X_{.j} \succcurlyeq_{1} Y_{.j}$ *for* $j = 1, 2$

Theorem

Let $d = 2$, *the following statements are equivalent:*
(i) $X \geqslant_{2\mathcal{U}_{SM}} Y$
(ii) $Y \succcurlyeq_{lo} X$ *and* $X_{.j} = Y_{.j}$ *for* $j = 1, 2$
(iii) $X \succcurlyeq_{uo} Y$ *and* $X_{.j} = Y_{.j}$ *for* $j = 1, 2$

See also Atkinson, Bourguignon (1987), Bourguignon, Chakravarty (2002), Athey (2000, 2002).

For $d > 2$ some of the equivalences may break.

Theorem

Let $d = 2$, *and* $X_{.j} = Y_{.j}$ *for* $j = 1, 2$ *the following statements are equivalent:*
(i) $X \geqslant_{2\mathcal{U}_{SM}} Y$; *(ii)* $Y \succcurlyeq_{lo} X$ *(iii)* $X \succcurlyeq_{uo} Y$ *(iv)* $Cov[f_1(X_{.1}), f_2(X_{.2})] \geq Cov[f_1(Y_{.1}), f_2(Y_{.2})]$ *for increasing functions* f_1, f_2; *(v)* $X \geqslant_{d_{\Delta_{\mathcal{M}}}} Y$.

One more result

Scarsini (J appl. Prob 1998)

Theorem

If $X_{.j} = Y_{.j}$ *for all* $j = 1, 2, .., d$ *then* $Y \geqslant_{d\mathcal{U}_{SM}} X$ *implies that* Xp *Lorenz dominates* Yp *for all* $p \in \mathbb{R}_+^d$.

A closer look at dependence between variables..

The dependence structure of a distribution can be represented by a Copula (a random vector uniformly distributed between [0, 1])
Consider for the Frechet class $\Gamma(F_{X_{.1}}, F_{X_{.2}}, ..., F_{X_{.d}})$ of d dimensional distributions with $F_{X_{.1}}, F_{X_{.2}}, ..., F_{X_{.d}}$ as marginals.
Given a $F_X \in \Gamma(F_{X_{.1}}, ..., F_{X_{.d}})$ there exist $C : [0, 1]^d \longrightarrow [0, 1]$ s.t. for all $\mathbf{x} \in \mathbb{R}^d$ $F_{\mathbf{X}}(\mathbf{x}) = C[F_{\mathbf{X}_{.1}}(x_1), F_{\mathbf{X}_{.2}}(x_2), ..., F_{\mathbf{X}_{.d}}(x_d)]$ and can be constructed if F_X is continuous as:

$$C[\mathbf{u}] := F_X[F_{x_{.1}}^{-1}(u_1); F_{x_{.2}}^{-1}(u_2); ..., F_{x_{.d}}^{-1}(u_d)] \quad \mathbf{u} \in [0, 1]^d$$

Theorem

If X and Y have a common copula then $X_{.j} \succcurlyeq_1 Y_{.j}$ for all $j = 1, 2, ...d$ implies $X \geqslant_{d\mathcal{U}_1} Y$.

Dominance in terms of the distributions of each attribute is sufficient to guarantee multivariate dominance!