On Lorenz Preorders and Opportunity Inequality in Finite Environments

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Theorem

For any $\mathbf{x} = (x_1, ..., x_n)$, $\mathbf{y} = (y_1, ..., y_n) \in \mathbb{R}^N$, *the following conditions are equivalent:*

(i) $\mathbf{y} \succcurlyeq^M \mathbf{x}$, *i.e.* $\sum_{h=1}^{k} (\mathbf{y}\downarrow)_h \geq \sum_{h=1}^{k} (\mathbf{x}\downarrow)_h$, $\quad h = 1, ..., n-1$, *and* $\sum_{h=1}^{n} (\mathbf{y}\downarrow)_h = \sum_{h=1}^{n} (\mathbf{x}\downarrow)_h$,

where, for any $\mathbf{u} = (u_1, ..., u_n)$, $\mathbf{u}\downarrow = (u_{\sigma(1)}, ..., u_{\sigma(n)})$, *with* $\sigma : N \longrightarrow N$ *a permutation such that* $\sigma(i) \geq \sigma(j)$, *entails* $u_i \leq u_j$;

(ii) $\mathbf{x}$ *can be derived from* $\mathbf{y}$ *through a finite sequence of transformations* $\mathbf{z}' = f(\mathbf{z})$ *of the following type:*

$$z_i' = z_i + \delta,\ z_j' = z_j - \delta \quad \text{with } j \leq i \text{ and } z_k' = z_k, \quad k \neq i, j \text{ and } \delta > 0$$

provided $\delta \leq (z_j - z_i)/2$;

(iii) $f(\mathbf{y}) \geq f(\mathbf{x})$ *holds for any* $f : \mathbb{A} \subset \mathbb{R}^N \rightarrow \mathbb{R}$ *of the following form: for each* $\mathbf{z} \in \mathbb{A}$, $f(\mathbf{z}) = \sum_{i=1}^{n} \varphi(z_i)$ *where* $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ *is a continuous convex function.*

- Opportunities rather than income levels are the proper targets of redistributive policies. Indeed, since individuals have different abilities to convert their resources into well-being, it is widely recognized that income may be utterly inadequate as an indicator of the opportunity for welfare they enjoy. (see Rawls (1971), Sen (1985, 2003), Roemer (1996)).
- Moreover, the desirability of reducing the disparity in the distribution of opportunities plays a prominent role in the platforms of many political parties and is a widely-accepted principle of distributive justice.

- Unfortunately, it is not clear *if* and *how* those (personal) characteristics/criteria might be aggregated. In other words, the analysis of opportunity in terms of inequality is complicated by the fact that an individual's opportunities are described by a set rather than by a scalar, as in the case with income or wealth inequality.
- Indeed, the assessment of inequality in resource allocation by means of Lorenz preorders is both well-established for univariate distributions and highly problematic for multivariate ones. The main reason for such a state of affairs is the following:
  - if the relevant variables are real-valued, the univariate case allows a natural *total* ordering of individual endowments, whereas any multivariate distribution, real-valued or otherwise, typically admits only *partial* rankings (e.g. dominance orderings) of the latter as natural and non-controversial.

So a **new problem** rapidly arises:

Problem

How to compare (and the rank) sets (of opportunities)?

Solution (Preference-for-flexibility)

Assume that there exists a nonempty set of alternatives X and an individual preference ordering R on X which, unless stated otherwise, is assumed to be linear. A relation $\succeq$ on the set of all nonempty and finite subsets $\mathcal{P}(X)$ of X (which are interpreted as opportunity sets or menus from which an agent can make a choice) is to be established.

Example (Kreps (Econ. 1979))

The indirect-utility ranking $\succeq_U$ of opportunity sets is defined by letting, for all $A;B \in \mathcal{P}(X)$

$$A \succeq_U B \text{ if and only if } \max(A)\, R \max(B)$$

That is, only the best elements according to R in the sets to be compared matter in establishing an ordering on X.

Kreps (Ec.1979) provides a characterization of the indirect-utility criterion for R not fixed. The axiom used in this characterization is the following extension-robustness condition. It requires that adding a set B that is at most as good as a given set A to A leads to a set that is indifferent to A itself.

Axiom

For all $A;B \in \mathcal{P}(X)$,

$$A \succeq B \rightarrow A \sim A \cup B.$$

Proposition

An ordering on X satisfies Extension Robustness if and only if there exists an ordering R on X such that is the indirect-utility ranking for the ordering R.

The indirect-utility criterion is based on the position that the quality of the final choice of the agent is all that matters, and the only reason other characteristics of an opportunity set might be of interest is that they may have instrumental value in achieving as high a level of well-being as possible.

(i) The way alternatives are formulated in economic models is often very restrictive, and they may not capture everything of value to an agent. In this case, utility is not an indicator of overall well-being but, rather, a measure of one aspect of well-being.

(ii) The ranking of opportunity sets should arguably only take into account the 'size' of the relevant set, without making any use of information about individual preferences which may be highly *unreliable*, *costly to acquire*, or both.

Solution (Freedom of choice)

We might want to rank opportunity sets in terms of the freedom of choice they offer: freedom of choice has a value that is independent of the amount of utility that may be generated by such freedom. Then, one can think of the volume of options figuring in the opportunity set as to be relevant. When the number of options is finite, the simplest way of assessing the volume or quantity of options available to the agent is to count how many options there are in the (opportunity) set.

Example (Pattanaik and Xu (1990))

They follow an axiomatic approach to the problem, and use three axioms to characterize a rule for ranking finite opportunity sets on the basis of their cardinalities. This cardinality-based ordering $\succeq_C$ X is defined by letting, for all $A;B \in \mathcal{P}(X)$

$$A \succeq_C B \text{ if and only if } |A| \geq |B| .$$

Axiom (Indifference Between No-Choice Situations (NC))

For all $x, y \in X$,

$$\{x\} \sim \{y\}.$$

Axiom (Simple Expansion Monotonicity (M))

For all distinct $x, y \in X$,

$$\{x, y\} \succ \{y\}.$$

Axiom (Strong Independence (IND))

For all $A;B \in \mathcal{P}(X)$ *and* $x \in X \setminus (A \cup B)$,

$$A \succeq B \text{ if and only if } A \cup \{x\} \succeq B \cup \{x\}.$$

Theorem

Suppose $\succeq$ is a complete preorder on $\mathcal{P}(X)$. Then, $\succeq$ satisfies NC, M, IND if and only if $\succeq = \succeq_C$.

- Critics to cardinality total preorder of opportunity sets:
  (i) However, the cardinality-based preorder is typically regarded as a trivial and uninteresting ranking of opportunity sets.
  (ii) Indeed, rejection of the cardinality ranking is quite obvious if some relevant evaluations of (or preferences on) opportunities are available.
  (iii) But the cardinality-based preorder is also typically rejected as a ranking of opportunity sets in terms of *freedom of choice*, namely when reliable and detailed preferential information is not available or is deemed to be not relevant.

Other two extra complications in comparing sets of opportunities

1. While income is a private good hence both excludable and rivalrous, opportunities may be non-rivalrous and possibly non-excludable according to the nature of the goods they are attached to. Thus, opportunities may be conceivably of a *private*, *public* or pure-public type.
2. It can be plausibly maintained that opportunities as opposed to the income levels are inherently *multidimensional* objects.

As a consequence of so-many complications, it was only with the seminal work of Kranich (Jet, 1996) that the question of how to rank different distributions of opportunities in terms of inequality they exhibit was first addressed.

- **Kranich** (Jet, 1996):

starts with a two-individual society and seeks to characterize a specific equality ordering on $\mathcal{P}(X) \times \mathcal{P}(X)$. Then, each element $(A_1, A_2) \in \mathcal{P}(X)^2$ represents a distribution of opportunity sets, where individual 1 has the opportunity set A_1 and individual 2 has the opportunity set A_2. For a two individual society, Kranich imposes the following axioms on an equality ordering $\succeq_e$ on $\mathcal{P}(X)^2$.

Axiom (Anonimity)

For all $(A_1, A_2) \in \mathcal{P}(X)^2$,

$$(A_1, A_2) \sim_e (A_2, A_1) .$$

Axiom (Monotonicity of Equality)

For all $A_1, A_2, A_3 \in \mathcal{P}(X)$ *such that* $A_1 \subseteq A_2 \subset A_3$,

$$(A_1, A_2) \succ_e (A_1, A_3) .$$

Axiom (Independence of Common Expansions)

For all $A_1, A_2, A_3 \in \mathcal{P}(X)$ *such that* $A_3 \cap (A_1 \cup A_2) = \varnothing$

$$(A_1 \cup A_3, A_2 \cup A_3) \sim_e (A_1, A_2) .$$

Axiom (Assimilation)

For all $(A_1, A_2) \in \mathcal{P}(X)^2$, *for all* $a \in A_1$, $b \in A_2$,
$c \in (X \setminus [(A_1 \setminus \{a\}) \cup (A_2 \setminus \{b\})])$,

$$((A_1 \setminus \{a\}) \cup \{c\}, (A_2 \setminus \{b\}) \cup \{c\}) \succeq_e (A_1, A_2).$$

Kranich shows that the only equality ordering $\succeq_e$on $\mathcal{P}(X)^2$ satisfying the above four axioms is the cardinality-difference ordering $\succeq_{CD}$defined as follows.

Definition

For all $(A_1, A_2), \left(A_1', A_2'\right) \in \mathcal{P}(X)^2$,

$$(A_1, A_2) \succeq_{CD} \left(A_1', A_2'\right) \iff ||A_1| - |A_2|| \geq \left|\left|A_1'\right| - \left|A_2'\right|\right|.$$

Kranich (Jet, 1996) relies on *differences* between *cardinalities* of opportunity sets in order to address the issue of ranking *profiles* of *non-rivalrous* opportunities in terms of inequality. In that connection, he provides a characterization of a class of indices of opportunity (in)equality that is strictly related to the class of generalized Gini inequality indices.

- **Herrero, Iturbe-Ormaetxe, and Nieto** (Mass, 1998)

For a two-person society, one can plausibly argue that the larger the number of options that are common between the opportunity sets of two individuals, the higher the degree of equality. In other words, Herrero, Iturbe-Ormaetxe, and Nieto (Mass, 1998) do rely on *total* preorders of opportunity sets, and provide characterizations of several egalitarian and utilitarian-like *total* preorders of opportunity profiles, including a few new criteria which emphasize the role of common opportunities (i.e. opportunities belonging to every component of the relevant profile). Similarly, Arlegi and Nieto (LGS1, 1999) offer axiomatizations of certain *total (in)equality* preorders of opportunity profiles which only depend on cardinality differences and/or the number of common opportunities.

Impossibility results

- **Ok and Kranich** (SCW, 1998)

They focus on the case of a two-individual society where, for each individual, the alternative opportunity sets are ranked on the basis of their cardinalities. In this framework, they prove an analogue of a basic theorem in the literature on the measurement of income inequality.

- They first introduce the notion of an equalizing transformation of a given pair of opportunity sets in their two-person society and also the notion of a Lorenz quasi-ordering on the set of pairs of opportunity sets.

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- They first introduce the notion of an equalizing transformation of a given pair of opportunity sets in their two-person society and also the notion of a Lorenz quasi-ordering on the set of pairs of opportunity sets.
- The main result of Ok and Kranich (SCW, 1998) shows that, in their assumed framework, one distribution of opportunity sets Lorenz dominates another distribution if and only if the first distribution can be reached from the second by a finite sequence of equalizing transformations and if and only if every inequality-averse social welfare functional ranks the first distribution higher than the second.

Let X denote the *finite* set of alternatives/opportunities, $N = \{1, ..., n\}$ the population of agents, and $\mathcal{P}(X)$ the power set of X, i.e. the set of its *opportunity sets*, with $\#X \geq 3$, in order to avoid trivial qualifications. We are interested in those opportunity rankings $(\mathcal{P}(X), \succcurlyeq)$ that arise whenever all the alternatives are "never bad".

Theorem (Ok and Kranich (1998))

If $(\mathcal{P}(X), \subset, \succcurlyeq_{\#})$ then for any pair of opportunity distributions $\mathbf{A}, \mathbf{B} \in (\mathcal{P}(X))^N$, $\mathbf{A} \prec \mathbf{B}$ iff $\mathbf{A}$ is reachable from $\mathbf{B}$ through a finite sequence of (suitably defined) Pigou-Dalton transfers or iff $f(\mathbf{A}) \leqslant f(\mathbf{B})$ for any (suitably defined) Schur-concave function f.

- **Ok** (Jet, 1997)

A general result due to Ok (Jet, 1997) has a pessimistic message regarding the possibility of measuring inequality in the distribution of opportunities. Ok formulates the counterpart of the fundmental concept of an equalizing transfer familiar in the literature on income distribution, and shows that the only ranking of opportunity sets that can serve as a basis of the notion of an equalizing transfer, as formulated by him, must be the cardinality-based ranking.

Indeed, the cardinality-based preorder is the sole **Strict Set-Inclusion Monotonic total preorder** which supports such a Lorenz-like preorder of opportunity distributions, namely:

Theorem (Ok (1997))

Let $(\mathcal{P}(X), \subset, \succcurlyeq)$ be such that $(\mathcal{P}(X), \succcurlyeq)$ is a Strict Set-Inclusion Monotonic totally preordered set, and for any pair of opportunity distributions $\mathbf{A}, \mathbf{B} \in (\mathcal{P}(X))^N$, $\mathbf{A} \prec \mathbf{B}$ iff $\mathbf{A}$ is reachable from $\mathbf{B}$ through a finite sequence of (suitably defined) Pigou-Dalton transfers or iff $f(\mathbf{A}) \leqslant f(\mathbf{B})$ for any (suitably defined) Schur-concave function f. Then $\succcurlyeq = \succcurlyeq_{\#}$.

Ok (Jet 1997)'s central result have a strong negative flavour, given the restrictive nature of the cardinality-based ranking of opportunity sets.

- From this review of the relevant literature, it appears that the majorization preorder of opportunity profiles is currently confined to a relatively marginal role and the prevailing interpretation of Ok's theorem may partly explain such an attitude.

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- From this review of the relevant literature, it appears that the majorization preorder of opportunity profiles is currently confined to a relatively marginal role and the prevailing interpretation of Ok's theorem may partly explain such an attitude.
- Indeed, the cardinality preorder of opportunity sets is commonly (and rightly) rejected as trivial. But then, if the cardinality preorder is the *only* one which supports an opportunity counterpart to the classic HLP theorem, it follows that the main result in Ok (Jet, 1997) is to be regarded as an '**impossibility theorem**' of sorts on majorization-consistent rankings of opportunity profiles (see also Barberà, Bossert and Pattanaik (2004) on this point).

Filtral preorders and opportunity inequality

- Such an interpretation should be firmly resisted: Ok's (1997) results only hold for *strictly inclusion-monotonic* (or 'contraction-consistent' inclusion-monotonic) rankings.
- We propose to relax this stringent requirement by focusing *on the entire class of inclusion-monotonic rankings*. Then, we show that, in this broader environment, it is after all *possible* to extend the celebrated HLP theorem to the measurement of opportunity inequality even starting from several preorders of opportunity sets which are different from the cardinality preorder.

- We propose to rely on a class of minimal extensions of the set-inclusion partial order whose members we shall refer to as *set-inclusion filtral preorders* (SIFPs).
- A SIFP on a (finite nonempty) set X of basic alternatives/opportunities amounts to an elementary way to augment the set-inclusion partial order with a *minimum opportunity-threshold*: under the threshold, opportunity sets are indifferent to each other and to the null opportunity set, while over the threshold the set-inclusion partial order is simply replicated.

Remark

- Also in a discrete setting namely when the resources to be allocated amount to a finite set of items/opportunities, it is by no means obvious if and how the non-controversial *set-inclusion partial preorder* might be extended to a *total* preorder of opportunity sets in order to define a Lorenz-like preorder of opportunity distributions amenable to characterizations via simple progressive Pigou-Dalton transfers as established by the classic Hardy-Littlewood-Polya theorem for real-valued (income) distributions.

A long digression

We are considering those Lorenz-style preorders of opportunity distributions which satisfy a counterpart of the foregoing HLP theorem.

- However, the Lorenz-based comparisons of univariate distributions are allowed by the *total* ordering induced by the perfect comparability of the individual incomes. On the contrary, individual endowments, namely multivariate distributions of personal goods/alternatives (hence opportunities), typically admit only *partial* non-controversial orderings.
- Therefore, a Lorenz preorder of opportunity distributions requires the preliminary definition of a *total* preorder on opportunity sets, an apparently controversial task.

- As a matter of fact, the problem of building up a Lorenz-like preorder, starting from a partial (pre)ordering in a finite setting, has not received yet in the literature the attention it deserves. The few exceptions include some works such as Hwang (1979), Lih (1982), Hwang and Rothblum (1993), which focus on Lorenz preorder when the set of population units is endowed with a **fixed partial order**.

Majorization on partial orders

- Hwang (Proc. Amer. Mathe. Soc, 1979) extends the classical concept of Lorenz (or dually majorization) preorder on a set of distributions to the case where the set of coordinates or equivalently of population units is partially ordered. His results rely, quite unexpectedly, on a classical theorem of Shapley on the existence of the core for every convex game and parallel the mentioned result of Muirhead on the equivalence between the Lorenz order and a Pigou-Dalton finite sequence of transfers.

- Lih (Siam J. of Agebr. Discr. Math., 1982) also extends the concept of majorization to the case of real-valued functions defined on a finite partially ordered set. More precisely, Lih defines the majorization preorder as follows: let $(P, \leq)$ denote a finite poset and Φ the set of all real-valued functions on $(P, \leq)$, if $\alpha, \beta \in \Phi$ then α majorizes β if, for any order filter U of $(P, \leq)$, $\alpha(U) \geq \beta(U)$ and $\alpha(P) = \beta(P)$ where, for any $\gamma \in \Phi$, $\gamma(U) = \sum\{\gamma(x) : x \in U\}$. In such a setting, he replicates the classical result of HLP reviewed above.

- Hwang and Rothblum (Mathem. Oper. Res, 1993) provide a further extension of majorization and Schur convexity with respect to partial orders over the coordinates of an Euclidean space.
  - They introduce the notion of 'pairwise connectedness' with respect to posets, which is actually a generalization of the Pigou-Dalton criterion of transfers, in order to achieve a characterization of Schur convexity (namely condition (*iii*) in Theorem of HLP above), with respect to partial orders for the case when every Schur convex function is neither necessarily symmetric (as in Lih (1982) and in the original work of Schur (1923)) nor asymmetric (as in Hwang (1979)).

- They also provide necessary and sufficient conditions for Schur convexity which rely on two-coordinate local properties of functions. That result implies that conclusions about local behavior of functions can be drawn without being forced to check every pair of coordinates on the (symmetric) domain of the function. Hence, their characterization of majorization via Schur convexity applies to a wider class of functions than those which are continuously differentiable.

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- In particular, Hwang and Rothblum extend the original Schur-(Ostrowski) theorem (see Marshall and Olkin (1979) chapter 3)). Such a generalized approach to majorization with a partial order of population units or dimensions might conceivably be extended to a finite environment.

Setting

We are interested in a class of opportunity rankings that arise whenever:

1. all the alternatives are "never bad": indeed, they are typically "good" but not necessarily so;
2. as a consequence of (1), the set-inclusion ordering is *weakly* respected but not strictly so, i.e. our rankings are set-inclusion monotonic (as opposed to strictly set-inclusion monotonic);
3. a minimum standard (threshold) is introduced such that any opportunity set which does not meet the corresponding standard is simply not acceptable i.e. is equivalent to the null set.

Thus, we focus on a preordered set $(\wp(X), \succcurlyeq)$ which extends the set-inclusion poset, namely $A \supseteq B$ entails $A \succcurlyeq B$ for all $A, B \in \wp(X)$.

Fact

A preordered set or preposet is a pair $(Y, \succcurlyeq)$ such that Y is a set and $\succcurlyeq$ is a reflexive and transitive binary relation on Y. A poset is a preposet $(Y, \succcurlyeq)$ such that $\succcurlyeq$ is antisymmetric.

In order to capture the notion of a threshold in this setting, we shall rely on the definition of an **order filter** *of a poset*. In fact, such an order filter collects all the elements of the poset which are greater than some member of a specified list of *noncomparable* elements, the so-called *generators*.

Definition ((Principal) Order Filters of a Poset)

Let $(Y, \succcurlyeq)$ be a non-empty poset and $\mathcal{B}$ an *antichain* of $(Y, \succcurlyeq)$, namely $\mathcal{B} \subseteq Y$ and for any $b_i, b_j \in \mathcal{B}$ if $b_i \neq b_j$ then *not* $b_i \succ b_j$. An order filter of $(Y, \succcurlyeq)$ with basis $\mathcal{B}$ is a set $F = F(\mathcal{B}) \subseteq Y$ such that

1. $\mathcal{B} \subseteq F$ and
2. for any $A, B \in Y$, if $A \in F$ and $B \succ A$ then $B \in F$.

Set-Inclusion Filtral Preorders

We use principal order filters of the set-inclusion poset $(\wp(X), \supseteq)$ to introduce a *filtral extension* of the latter. This amounts to enriching $(\wp(X), \supseteq)$ with a suitable threshold, which, in turn, corresponds to the requirement of a minimum level of freedom, i.e. an opportunity poverty line of sorts. To repeat, below the threshold the available amount of individual *freedom of choice* is deemed to be *not* acceptable.

Example

Think e.g. of a citizen that has access to any newspaper she likes to read and enjoys freedom of speech but is deprived of voting rights.

All this can be embodied in the following:

Definition (Set-Inclusion (Principal) Filtral Preorders (SIFPs))

For any (principal) order filter F of $(\wp(X), \supseteq)$ the F-generated *set-inclusion (principal) filtral preorder (SIFP)* is the binary relation $\succcurlyeq_F$on $\wp(X)$ defined as follows: for any $A, B \in \wp(X)$, $A \succcurlyeq_F B$ if and only if $A \supseteq B$ or $B \notin F$.

- Notice that, under the extremal or degenerate cases $F = \wp(X)$ and $F = \varnothing$, $(\wp(X), \succcurlyeq_F)$ reduces to the set-inclusion order and the degenerate total preorder consisting of a single indifference class, respectively.

As said, the main aim of the present paper is to propose a *SIFP*-based method of ranking profiles of opportunity sets in terms of opportunity inequality. In order to accomplish the foregoing task we have to introduce the following:

Definition

Let F be a (principal) order filter of $(\wp(X), \supseteq)$ and $\succcurlyeq_F$ the (principal) SIFP induced by F. Then, the $\succcurlyeq_F$*-induced height function*

$$h_{\succcurlyeq_F} : \wp(X) \rightarrow \mathbb{Z}_+$$

is defined as follows: for any $A \subseteq X$:

$$h_{\succcurlyeq_F}(A) = \max \left\{ \begin{array}{c} \#\mathcal{C} : \mathcal{C} \text{ is a } \succcurlyeq_F \text{-chain, such that} \\ A \in \mathcal{C} \text{ and } A \succ_F B \text{ for any } B \in \mathcal{C} \setminus \{A\} \end{array} \right\}.$$

- In words, the height function assigns to each opportunity set A a non-negative number, namely the size of the longest strictly ascending chain having A as its maximum.

Fact

Recall that a chain of a preordered set $(Y, \succcurlyeq)$ is a subset $Z \subseteq Y$ such that $(Z, \succcurlyeq)$ is a totally preordered set.

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- We start by adjoining a (principal filter-induced) threshold to the set inclusion ordering of opportunity sets, which, of course, provides a (principal) filtral opportunity preorder.
- Then, we consider the resulting heights for the opportunity profiles under consideration.
- Next we apply the majorization preorder of Theorem of HLP above to the set of height profiles we obtain. Such a preorder induces in a natural way a preorder on opportunity profiles. This is an inequality ranking of opportunity profiles, which is a counterpart of the (*dual of*) Lorenz ranking of income distributions.

Thus the procedure we propose can be summarized as follows:

1. take a principal *SIFP* $(\wp(X), \succcurlyeq_F)$ on $(\wp(X), \supseteq)$;
2. consider the $\succcurlyeq_F$-induced height function $h_{\succcurlyeq_F}$;
3. use the majorization preorder on height profiles in order to define a generalized $\succcurlyeq_F$-induced majorization preorder $\succcurlyeq_F^M$ on the set $(\wp(X))^N$ of N-profiles of opportunity sets.

Analitically, we denote the set of all admissible opportunity profiles for population N as $(\wp(X))^N$. With a slight abuse of notation, we denote by $\mathbf{A} = (A_i)_{i \in N}$ a generic opportunity profile. Hence, for any $i \in \{1, ..., N\}$, A_i represents the set of opportunities allotted to individual (or group) i according to $\mathbf{A}$.

- Thus we focus on a quite general domain of opportunity profiles. One characteristic of $((\wp(X))^N, \succcurlyeq_F^M)$ is that it works by mapping the space of opportunity profiles into a set of integer points in $\mathbb{Z}_+^N$, i.e. the space of height vectors. This set will of course depend on the relevant principal order filter F and is therefore denoted as the (height) *span* of $\succcurlyeq_F$, written $H_{\succcurlyeq_F}$.

Therefore, we proceed to define an opportunity-profile-counterpart of the majorization preorder as defined in the Theorem of HLP (i):

Definition

Let $\mathbf{A}, \mathbf{B} \in (\wp(X))^N$ be two opportunity profiles, F a principal order filter of $(\wp(X), \supseteq)$, $\succcurlyeq_F$the corresponding set-inclusion (principal) filtral preorder (SIFP) on $\wp(X)$, and $h_{\succcurlyeq_F}$the $\succcurlyeq_F$-induced height function on $\wp(X)$. Then $\mathbf{A}$ majorizes $\mathbf{B}$, denoted $\mathbf{A} \succcurlyeq_F^M \mathbf{B}$, if

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i \tag{1}$$

and

$$\left(h_{\succcurlyeq_F}(A_1), ..., h_{\succcurlyeq_F}(A_n)\right) \succcurlyeq^M \left(h_{\succcurlyeq_F}(B_1), ..., h_{\succcurlyeq_F}(B_n)\right), \tag{2}$$

where $\succcurlyeq^M$denotes the majorization preorder as defined above under Theorem 1.i. We denote by $\succ_F^M$ the asymmetric component of $\succcurlyeq_F^M$.

Warning

- The restriction to principal order filters does not come entirely for free. Indeed, opportunities are attached to resources which may be either rivalrous or non-rivalrous, and excludable or not. However, if the threshold amounts to a *unique* minimal opportunity set *and* such a minimal opportunity set includes *one or more rivalrous opportunities* (e.g. a *private* opportunity), then within our framework for each possible allocation of opportunities there can be at most *one* population unit that stands over or above the threshold. It is easily checked that, under this case, the majorization preorder reduces to its symmetric component.Hence, our main result, which provides two equivalent representations of the *asymmetric* component of the majorization preorder, reduces to a trivially true statement.

- On the other hand, and independently of the number of filter's generators, if *every* opportunity in any minimal opportunity set happens to be *non-excludable* (e.g. a pure public opportunity for the relevant population), then, by construction, all the population units stand over the opportunity threshold, which has therefore no tangible effect within our model.

- Thus, a sensible **interpretation of our model** requires that *all* opportunities in the unique generator of the relevant principal order filter be *non-rivalrous,* and at least *some* of them be *excludable*. The most straightforward way to ensure that, without any further ado, is assuming that *all opportunities in the basic set* X *are both non-rivalrous and excludable,* that is in fact the interpretation we suggest.

Inequality ranking of opportunity profiles

Now, in order to replicate the basic findings of the literature on income inequality in the opportunity-profile setting, we must provide a suitable reformulation of the Pigou-Dalton transfer principle:

Definition

A transfer operator on $(\wp(X))^N$ is a nonempty correspondence $\Im : (\wp(X))^N \rightrightarrows (\wp(X))^N$ such that $\forall (\mathbf{A}, \mathbf{B}) \in (\wp(X))^N \times \Im\left((\wp(X))^N\right) : \left[\bigcup_i A_i = \bigcup_i B_i\right]$.

Next, we define a notion of simple or minimal bilateral transfer:

Definition

Let $\mathbf{A}, \mathbf{B} \in (\wp(X))^N$ be two opportunity profiles, F a principal order filter of $(\wp(X), \supseteq)$ and $i, j \in N$ such that $A_j \succ_F A_i$, $x \in A_j \setminus A_i$,

$$B_j = A_j \setminus \{x\}, \; B_i = A_i \cup \{x\}, \; B_k = A_k \quad k \neq i, j$$

Then $\mathbf{B}$ is said to arise from $\mathbf{A}$ through a *simple* (i.e. bilateral and minimal) *transfer* (from j to i). A transfer operator $\Im$ on $(\wp(X))^N$ shall be said *simple* if for any $\mathbf{A}, \mathbf{B}$ such that $\mathbf{B} \in \Im(\mathbf{A})$, $\mathbf{B}$ arises from $\mathbf{A}$ through a simple transfer.

By analogy with the Pigou-Dalton principle, we also require that transfers of opportunities be not large enough to reverse the relative positions of the donor and recipient. This is the rationale of the next definitions, namely:

Definition

A transfer operator $\Im$ is said to be:

i) *weakly rank-monotonic* w.r.t $\succcurlyeq_F$ if and only if it does not cause height-reversals i.e. for any $\mathbf{A}, \mathbf{B} \in (\wp(X))^N$ and any $i, j \in N$,

$$\text{if } \mathbf{B} \in \Im(\mathbf{A}), B_i \neq A_i, B_j \neq A_j \text{ and } h_{\succcurlyeq_F}(A_j) \geq h_{\succcurlyeq_F}(A_i)$$

$$\text{then } h_{\succcurlyeq_F}(B_j) \geq h_{\succcurlyeq_F}(B_i);$$

ii) *weakly progressive* w.r.t. $\succcurlyeq_F$ if and only if for any $\mathbf{A}, \mathbf{B} \in (\wp(X))^N$:

$$\text{if } \mathbf{B} \in \Im(\mathbf{A}),\ B_i \supset A_i \text{ and } A_j \supset B_j$$

$$\text{then } h_{\succcurlyeq_F}(A_j) \geq h_{\succcurlyeq_F}(A_i).$$

iii) *weakly-equalizing*, or *Daltonian*, w.r.t. $\succcurlyeq_F$ if it is simple, weakly rank-monotonic w.r.t. $\succcurlyeq_F$ and weakly progressive w.r.t. $\succcurlyeq_F$.

Let us now proceed in our search for a SIFP-counterpart of HLP's Theorem. In order to pursue this aim, we have to focus on the class of real-valued functions which preserve SIFP-induced majorization preorders.

Definition (Real-valued $\succcurlyeq_F^M$-monotonic functions)

Let F be an order filter of $(\wp(X), \supseteq)$ and $\succcurlyeq_F^M$ the majorization preorder on $(\wp(X))^N$ induced by SIFP $\succcurlyeq_F$ as defined above. Then a real-valued function

$$f : (\wp(X))^N \longrightarrow \mathbb{R}$$

is $\succcurlyeq_F^M$-*monotonic* on domain $D \subseteq (\wp(X))^N$ if and only if for any $\mathbf{A}, \mathbf{B} \in D$

$$f(\mathbf{A}) \geq f(\mathbf{B}) \qquad \text{whenever } \mathbf{A} \succ_F^M \mathbf{B}.$$

Theorem (Savaglio and Vannucci (Jet, 2007))

Let F be a principal order filter of $(\wp(X), \supseteq)$, and $\mathbf{A}, \mathbf{B} \in (\wp(X))^N$ two opportunity profiles such that $\{h_{\succcurlyeq_F}(\mathbf{A}), h_{\succcurlyeq_F}(\mathbf{B})\} \subseteq H^+_{\succcurlyeq_F}$. Then, the following statements are equivalent:

1. $\mathbf{A} \succ^M_F \mathbf{B}$;
2. *There exist a $\succcurlyeq_F$-Daltonian transfer operator $\Im$ and a positive integer k such that* $\mathbf{B} \in \Im^{(k)}(\mathbf{A})$
3. $$(\varphi \circ h_{\succcurlyeq_F})(\mathbf{A}) \geq (\varphi \circ h_{\succcurlyeq_F})(\mathbf{B})$$
   for any $\varphi : \mathbb{Z}^N_+ \to \mathbb{R}$ such that $\varphi \circ h_{\succcurlyeq_F}$ is a $\succcurlyeq^M_F$-monotonic function on $[(H^+_{\succcurlyeq_F})^{-1}] \downarrow$.

- Thus, the foregoing Theorem is an opportunity-profile counterpart to the HLP theorem on inequality measurement as required.

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- Thus, the foregoing Theorem is an opportunity-profile counterpart to the HLP theorem on inequality measurement as required.
- It should be noticed that by taking $F = \wp(X)$ Theorem 2 specializes to a version of the similar characterization result of Ok and Kranich (1998).

"If we knew what it was we were doing, it would not be called research, would it?"

Albert Einstein (1879-1955)

A **bridge** between two research trends

Reliance on the set-inclusion order as implied by SIFPs, however, is only satisfactory when at least some of the relevant resources are **public**, or at least **non-rival** goods. Savaglio and Vannucci (2007) suggest one way to confirm the foregoing result while avoiding such a disturbing restriction.

- If individual endowments are modelled via multisets rather than set, then the items in the basic set X could be rivalrous and excludable objects, namely as **pure private goods**.
- Thus, the very same problem considered above can be addressed starting from the strict dominance order for *multisets*, as augmented with a threshold, a sort of multidimensional (opportunity) poverty line below which each opportunity set is indifferent to the null set.

Fact

A finite multiset on X is a function $m : X \to \mathbb{Z}_+$ such that $\sum_{x \in X} m(x) < \infty$. A partition of multiset m -or multipartion of m- on population N is a profile $\mathbf{m} = \{m_i\}_{i \in N}$ of multisets on X, such that for any $x \in X$: $\sum_{i \in N} m_i(x) = m(x)$.

- A partition of multisets, or *multipartition*, is a mathematical notion that mimics a multivariate distribution and that can be represented as a rectangular matrix

$$\begin{array}{ccl} & x \qquad y \quad \ldots \quad w \;\ldots \qquad z & \longleftarrow \text{goods} \\ \begin{array}{c}\text{people} \\ \downarrow\end{array} & \overbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}^{} & \\ \mathbf{m} = & \begin{bmatrix} m_1(x) & m_1(y) & \ldots & m_1(z) \\ . & . & & . \\ . & . & m_i(w) & . \\ . & . & & . \\ . & . & . & . \\ m_n(x) & \ldots & . & m_n(z) \end{bmatrix} & \end{array} \tag{3}$$

Problem

"Given two distribution matrices $\mathbf{m}$ *and* $\mathbf{m}'$*, which one contains the lower level of disparity?"*.

-

To answer the question, we generalize some suitable unidimensional dominance criteria to the multidimensional case. In particular, we generalize, the notion of Lorenz preorder to that of Lorenz preorder of partitions of finite multisets, defined with the reference to a preorder of sets of goods as induced by strict dominance and augmented with a threshold.

- Then, in order to proceed with our analysis, let M_X be the set of all multisets on X and define the natural componentwise (strict) order $>$ on M_X as follows: for any $m, m' \in M_X$, $m > m'$ if and only if $m(x) > m'(x)$ for any $x \in X$. In particular, for any $m^* \in M_X$, we may consider the subposet $\mathcal{M}_{m^*} = (M_{X,m^*}, >)$ of the poset $\mathcal{M} = (M_X, >)$, where $M_{X,m^*} = \{m \in M_X : m > m^* \text{ or } m = m^*\}$.

Definition (Majorization)

Let $\mathbf{m}, \mathbf{m}' \in \Pi_m^N$ be two profiles of individual endowments of goods, F an order filter of $(M_{X,m^*}, >)$, $\succcurlyeq_F$ the corresponding filtral preorder on M_{X,m^*}, and $h_{\succcurlyeq_F}$ the $\succcurlyeq_F$-induced height function on M_{X,m^*}. Then, we say that $\mathbf{m}$ majorizes $\mathbf{m}'$, denoted $\mathbf{m} \succcurlyeq_F^{maj} \mathbf{m}'$, if:

$$h_{\succcurlyeq_F}(\mathbf{m}) = \left(h_{\succcurlyeq_F}(m_1), ..., h_{\succcurlyeq_F}(m_n)\right) \succcurlyeq^{maj} \left(h_{\succcurlyeq_F}(m_1'), ..., h_{\succcurlyeq_F}(m_n')\right) = h_{\succcurlyeq_F}$$

namely:

$$\sum_{i=1}^{k} h_{\succcurlyeq_F}(m_i) \geqslant \sum_{i=1}^{k} h_{\succcurlyeq_F}(m_i') \qquad k = 1, ..., n-1,$$

$$\text{and } \sum_{i=1}^{n} h_{\succcurlyeq_F}(m_i) = \sum_{i=1}^{n} h_{\succcurlyeq_F}(m_i'),$$

whenever the height vectors are arranged in non-increasing order.

Let us state the notion of *transfer* with respect to height-extensions of DFPs, by first defining a *transfer operator* as follows:

Definition

A transfer operator on Π_m^N is a nonempty correspondence $\Im : \Pi_m^N \rightrightarrows \Pi_m^N$ such that

$$\forall \left(\mathbf{m}, \mathbf{m}'\right) \in \Pi_m^N \times \Pi_m^N, \ \mathbf{m}' \in \Im\left(\mathbf{m}\right).$$

Then, a transfer operator is a transformation which leaves the set of all total alternatives/goods in $\mathbf{m}$ and $\mathbf{m}'$ unchanged. Next, we define a notion of *minimal* transfer as:

Definition

Let $\mathbf{m}, \mathbf{m}' \in \Pi_m^N$ be two profiles of individual endowments of goods, F a principal order filter of $\left(M_{X,m^*}, >\right)$ with basis $\mathcal{B}_F = \{b\}$ and $i, j \in N$ such that $m_i \succ_F m_j$, $h_{\succcurlyeq_F}\left(m_i\right) > h_{\succcurlyeq_F}\left(m_j\right) + 1$ such that:

$$\begin{aligned} m_i'\left(x\right) &= m_i\left(x\right) - 1 \text{ for any } x \in X, \quad m_j'\left(x\right) = m_j\left(x\right) + 1 \text{ for any } x \\ \text{and } m_l'\left(x^*\right) &= m_l\left(x^*\right) \text{ for any } l \neq i, j, \text{ and } x^* \in X \end{aligned}$$

By analogy with the Pigou-Dalton principle, we also require:

Definition

A transfer operator $\Im$ is said to be:

(i) *weakly rank-monotonic* w.r.t $\succcurlyeq_F$ if and only if it does not cause height-reversals i.e. for any $\mathbf{m}, \mathbf{m}' \in \Pi_m^N$ and any $i, j \in N$, if

$$\mathbf{m}' \in \Im(\mathbf{m}), \;\; m_i' \neq m_i, \;\; m_j' \neq m_j$$

and $h_{\succcurlyeq_F}(m_i) \geqslant h_{\succcurlyeq_F}(m_j)$ then $h_{\succcurlyeq_F}(m_i') \geqslant h_{\succcurlyeq_F}(m_j')$.

(ii) *weakly progressive* w.r.t. $\succcurlyeq_F$ if and only if for any $\mathbf{m}, \mathbf{m}' \in \Pi_m^N$:

$$\mathbf{m}' \in \Im(\mathbf{m}), \; m_i' > m_i \text{ and } m_j > m_j'$$

entails that $h_{\succcurlyeq_F}(m_i) \geqslant h_{\succcurlyeq_F}(m_j)$.

(iii) *weakly-equalizing* w.r.t. $\succcurlyeq_F$ if it is both weakly rank-monotonic w.r.t. $\succcurlyeq_F$and weakly progressive w.r.t. $\succcurlyeq_F$.

Moreover, in order to pursue our search for a DFP-counterpart of the HLP's celebrated result, we have to focus on the class of real-valued functions which preserve DFP-induced majorization preorders.

Definition (real-valued $\succcurlyeq_F^{maj}$-isotonic functions)

Let F be an order filter of $(M_{X,m^*}, >)$ and $\succcurlyeq_F^{maj}$ the majorization preorder on Π_m^N induced by the DFP $\succcurlyeq_F$ as defined above. Then a real-valued function

$$f : \Pi_m^N \longrightarrow \mathbb{R}$$

is *isotonic* (wrt $\succcurlyeq_F^{maj}$) on domain $D \subseteq \Pi_m^N$ if and only if for any $\mathbf{m}, \mathbf{m}' \in D$

$$f(\mathbf{m}) \geqslant f(\mathbf{m}') \qquad \text{whenever } \mathbf{m} \succ_F^{maj} \mathbf{m}'.$$

Finally, the use of the foregoing Definitions is clarified in the following:

Example

Let us suppose that the set of available goods X is composed of six copies of good x and ten copies of good y, (i.e. $m(x) = 6$ and $m(y) = 10$), distributed over a population of three agents $\{i, j, l\}$ in order to get a partition of multiset m, namely the multiprofile:

$$\mathbf{m} = \begin{array}{c c} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} i \\ j \\ l \end{array} & \begin{pmatrix} 5 & 6 \\ 1 & 2 \\ 0 & 2 \end{pmatrix} \end{array}$$

If we consider as the *basis* of the filter $\mathcal{B}_F = \{b_1, b_2\}$, where $b_1 = (b_1(x)) = 1$ and $b_2 = (b_2(y)) = 1$, then the corresponding filter-induced height function will be tantamount to $h_{\succcurlyeq_F}(\mathbf{m}) = (5, 1, 0)$.

Example (Cont.)

Thus, suppose that a transfer takes place from richer i to poorer l in order to get the new multidimensional distribution:

$$\mathbf{m}^{'} = \begin{array}{cc} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} i \\ j \\ l \end{array} & \begin{pmatrix} 4 & 5 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \end{array}$$

and the corresponding $h_{\succcurlyeq_F}\left(\mathbf{m}^{'}\right) = (4, 1, 1)$. Hence, it is obvious that $\mathbf{m} \succcurlyeq_F^{maj} \mathbf{m}'$ and that $f(\mathbf{m}) \geqslant f(\mathbf{m}')$ where f is, for example, a function that simply sums the value of the heights of the multipartitions. On the contrary, if $\mathcal{B}_F = \{b_1,\ b_2\} = (1, 3)$, and the same transfer takes place in $\mathbf{m}$, we now have that $h_{\succcurlyeq_F}(\mathbf{m}) = (4, 0, 0)$ and $h_{\succcurlyeq_F}\left(\mathbf{m}^{'}\right) = (3, 0, 0)$, with corresponding net loss of height mass. It is worth noticing here how a careful check that numbers always vary according to transfers of goods is often required.

A HLP Theorem for finite multipartitions

Theorem (Savaglio and Vannucci (2007))

Let F be a principal order filter of $(M_{X,m^*}, >)$, and $\mathbf{m}, \mathbf{m}' \in \Pi_m^N$ two opportunity profiles such that $h_{\succcurlyeq_F}(\mathbf{m}), h_{\succcurlyeq_F}(\mathbf{m}') \in H^+_{\succcurlyeq_F}$. Then, the following statements are equivalent:

1. $\mathbf{m} \succ_F^{maj} \mathbf{m}'$;
2. *There exist a $\succcurlyeq_F$-weakly equalizing transfer operator $\Im$ and a positive integer k such that* $\mathbf{m}' \in \Im^{(k)}(\mathbf{m})$
3. $$(\varphi \circ h_{\succcurlyeq_F})(\mathbf{m}) \geqslant (\varphi \circ h_{\succcurlyeq_F})(m')$$
   for any $\varphi : \mathbb{Z}_+^N \to \mathbb{R}$ such that $\varphi \circ h_{\succcurlyeq_F}$ is a $\succ_F^{maj}$-isotonic function on $[(H^+_{\succcurlyeq_F})^{-1}]$.

Conclusion

- To conclude, the componentwise strict dominance preorder of vectors, representing the assignment of the goods to the agents, supports a multipartition counterpart to the celebrated HLP Theorem. In a sense, Savaglio and Vannucci (w.p. 2007) provide a somewhat optimistic answer to the question: "*A lost paradise?*", raised by Trannoy (2006) and concerning the *possibility* of finding again "*the miracle of the HLP theorem*" in the multidimensional context.

Conclusion

- To conclude, the componentwise strict dominance preorder of vectors, representing the assignment of the goods to the agents, supports a multipartition counterpart to the celebrated HLP Theorem. In a sense, Savaglio and Vannucci (w.p. 2007) provide a somewhat optimistic answer to the question: "*A lost paradise?*", raised by Trannoy (2006) and concerning the *possibility* of finding again "*the miracle of the HLP theorem*" in the multidimensional context.
- The relevance of the foregoing results relies on the fact that the DFP-approach is conducive to a majorization preorder of multiprofiles of goods that extends the classic unidimensional analysis of income inequality to a multivariate context.

- Since the comparison of multidimensional distributions typically admits only a non-total preorder of individual endowments, we have suggested the possibility to rely on height-based total extensions in order to reproduce some relevant parts of the theory of majorization (or, dually, Lorenz) preorders. Indeed, we have shown that the componentwise strict preorders of vectors, representing the assignment of the goods to the agents, support a multipartition counterpart to the celebreted HLP Theorem.

- Since the comparison of multidimensional distributions typically admits only a non-total preorder of individual endowments, we have suggested the possibility to rely on height-based total extensions in order to reproduce some relevant parts of the theory of majorization (or, dually, Lorenz) preorders. Indeed, we have shown that the componentwise strict preorders of vectors, representing the assignment of the goods to the agents, support a multipartition counterpart to the celebreted HLP Theorem.
- Our answer to the question: "*A lost paradise?*", posed by Trannoy (2006) and concerning the *impossibility* of finding again "*the miracle of the HLP theorem*" in the multidimensional context does not come totally for free.

- We first needed to use a two-steps procedure in order to compare rectangular matrices, representing the disparity of a population of N individuals distinguished for several attributes, namely multivariate ditributions of goods. Then, we adopted a very restricted version of the Pigou-Dalton principle of transfers to define a distributive profile as less even than another one.

- We first needed to use a two-steps procedure in order to compare rectangular matrices, representing the disparity of a population of N individuals distinguished for several attributes, namely multivariate ditributions of goods. Then, we adopted a very restricted version of the Pigou-Dalton principle of transfers to define a distributive profile as less even than another one.
- Although our work represents a new fruitful approach to the analysis of multidimensional inequality, much more remains to be discovered, at least on the problem to compare our solution to the issue of building up a Lorenz preorder of multivariate distributions with the main results on matrix majorization existing in economic literature, but this task is best left as a possible topic for further research.