

# Aggregation of preferences under uncertainty

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# Risk and Uncertainty

## Luce & Raiffa, 1957

- Certainty: each action is known to lead invariably to a specific outcome
- Risk: each action leads to one of a set of possible specific outcomes, each outcome occurring with a known probability. The probabilities are assumed to be known to the decision maker
- Uncertainty: each action has as its consequence a set of possible specific outcomes, but the probability of these outcomes are completely unknown or are not even meaningful (p.13)

# The Aggregation Problem

$$(x_E, E; x_{\bar{E}}, \bar{E})$$

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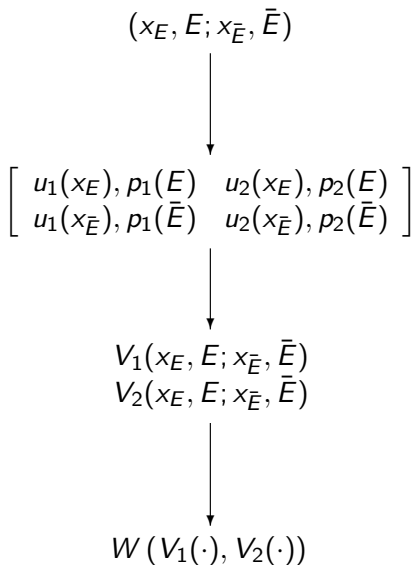
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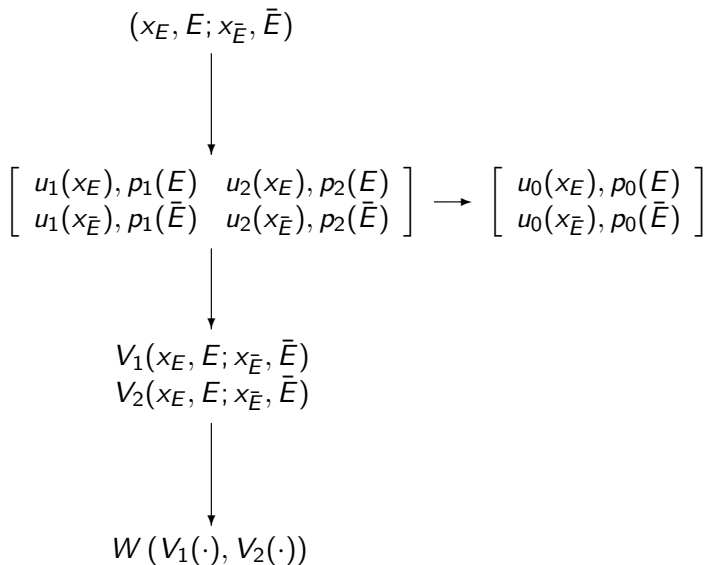
$$V_1(x_E, E; x_{\bar{E}}, \bar{E})$$

$$V_2(x_E, E; x_{\bar{E}}, \bar{E})$$

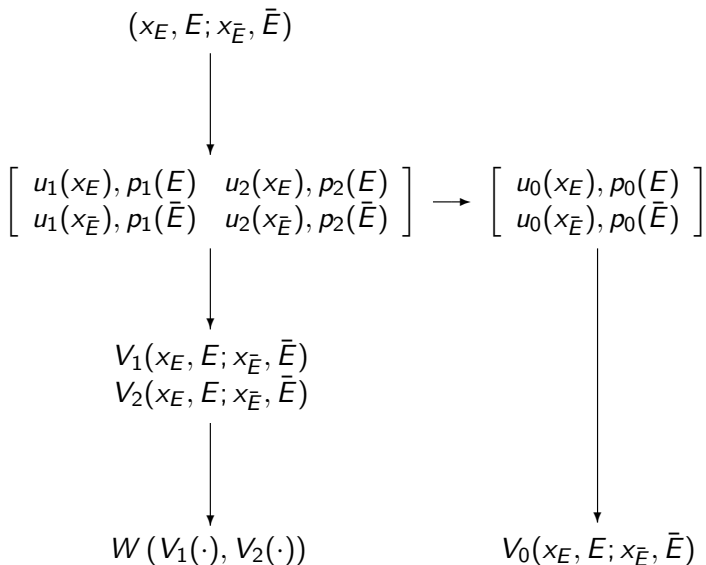
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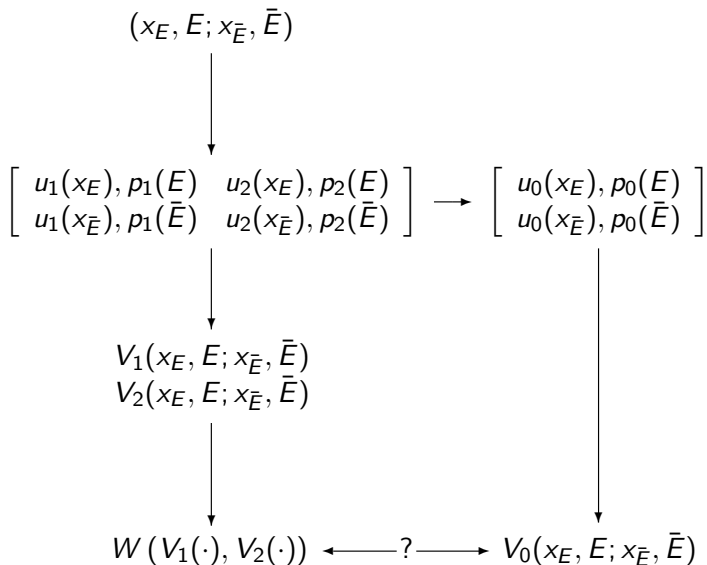


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# The aggregation problem

Aggregating  $n$  preferences into one that:

- 1 satisfies the same “rationality” requirements as individuals’ preferences
- 2 is non dictatorial
- 3 does not provoke unanimous opposition

# Road map

- 1 The vNM case: Harsanyi's aggregation theorem
- 2 The Subjective Expected Utility case
- 3 Uncertainty: the (almost) general case

# Setup

- $N' = \{1, \dots, n\}$  agents,  $N = \{0\} \cup N'$  where 0 = "society"
- $X$  (sure) social alternatives
- $\mathcal{L} = \left\{ p : X \rightarrow [0, 1] \mid \begin{array}{l} \#\{x \mid p(x) > 0\} < \infty \\ \sum_{x \in X} p(x) = 1 \end{array} \right\}$  social lotteries

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- individuals and society satisfy vNM axioms: for all  $i \in N$ , there exists  $u_i : X \rightarrow \mathbb{R}$  such that:

$$p \succsim_i q \Leftrightarrow \sum_{x \in X} p(x) u_i(x) \geq \sum_{x \in X} q(x) u_i(x)$$

Moreover,  $u_i$  is unique up to an increasing affine transformation.

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# Axioms

Weak Pareto (WP)

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## Independent Prospect (IP)

For all  $k \in N'$ , there exist  $p_k$  and  $q_k$  such that:

$$p_k \succ_k q_k \text{ and } p_k \sim_i q_k, \forall i \in N' \setminus \{k\}$$



# Harsanyi's aggregation theorem

## Theorem

Assume that  $\succsim_i$  is represented by a vNM function  $u_i$  ( $i \in N$ ) and (IP) is satisfied. Then (WP) holds iff there exist unique  $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}_+^n \setminus \{0_n\}$ ,  $\mu \in \mathbb{R}$  such that:

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## Remark

In Harsanyi's original (1955) theorem:

- Pareto indifference: sign of coefficients undetermined
- Independent Prospect not assumed: coefficients not unique

# Proof (sketch)

## Lemma 1

Under vNM, (IP) implies that there exist  $p^*$  and  $p_*$  such that:

$$p^* \succ_i q_*, \forall i \in N' \quad (\text{MA})$$

[◀ Proof](#)

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### Lemma 2 (De Meyer & Mongin, 1995)

Let  $X \neq \emptyset$  and  $F = (f_0, f_1, \dots, f_n) : X \rightarrow \mathbb{R}^{n+1}$ . If  $K = F(X)$  is convex and (WP) and (MA) hold, then there exist  $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}_+^n \setminus \{0_n\}$ ,  $\mu \in \mathbb{R}$  such that:

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Under the assumptions of Lemma 2, (IP) implies unicity of the coefficients.

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### Remark

*This theorem gave rise to a substantial body of work and often passionate debates. For a survey, see Sen (1986) and Weymark (1991). These debates concern:*

- *the interpretation of the theorem* ▶ more
- *its normative appeal* ▶ more

# Subjective Expected Utility

## Anscombe-Aumann setup

- $S$  finite set of states of nature
- $X$  social outcomes
- $Y$  simple probability distributions over  $X$  (roulette lotteries)
- $\mathcal{A} = \{f : S \rightarrow X\}$  acts (horse lotteries)
- $\mathcal{A}$  is a mixture space:  $(\alpha f + (1 - \alpha)g)(s) = \alpha f(s) + (1 - \alpha)g(s)$

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## Subjective Expected Utility

individuals and society satisfy SEU axioms: for all  $i$  there exist a probability measure  $p_i$  on  $S$  and a non-constant vNM function  $u_i : Y \rightarrow \mathbb{R}$  such that:

$$f \succsim_i g \Leftrightarrow \sum_{s \in S} p_i(s) u_i(f(s)) \geq \sum_{s \in S} p_i(s) u_i(g(s))$$

Moreover,  $p_i$  is unique,  $u_i$  is unique up to an increasing affine transform.



## Aggregation of SEU (Mongin, 1998)

### Theorem

Assume that  $\succsim_i$  is represented by a SEU function with utility  $u_i$  and beliefs  $p_i$  ( $i \in N$ ) and (IP) is satisfied. Then (WP) iff there exist unique  $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}_+^n \setminus \{0_n\}$ ,  $\mu \in \mathbb{R}$  such that for  $f \in \mathcal{A}$  :

$$\sum_s p_0(s) u_0(f(s)) = \sum_{i \in N'} \lambda_i \left( \sum_s p_i(s) u_i(f(s)) \right) + \mu.$$

Moreover,  $p_j = p_k = p_0$  for all  $j, k \in J = \{i \in N' \mid \lambda_i \neq 0\}$ .

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Wlog, pick  $y \in Y$  and let  $u_i(y) = 0, \forall i \in N'$ .

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## Second part of the theorem

By the first part:

- $p_0(s)u_0 = \sum_{i \in N'} \lambda_i p_i(s)u_i, \forall s \in S$  (acts in  $\mathcal{A}_{y,s}$ )

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$u_i$  ( $i \in J$ ) aff. indep.  $\Rightarrow p_0(s) = p_i(s), \forall i \in J, s \in S$

## What can we do?

- Relaxing Pareto: Gilboa, Samet and Schmeidler (2004)
- Allowing for less restrictive preferences
- After all, SEU is very special: it imposes **uncertainty neutrality**



## Uncertainty aversion

Ellsberg's paradox

90 balls in urn: 30 red, and 60 blue and yellow

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	<i>R</i>	<i>Y</i>	<i>B</i>
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<i>g</i> <sub>1</sub>	0	1	0
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## Inconsistent with SEU

- $f \succ g_1 \Rightarrow \Pr(Y) < \frac{1}{3}$
- $h \succ g_2 \Rightarrow \frac{2}{3} > \frac{1}{3} + \Pr(B) = \frac{1}{3} + (\frac{2}{3} - \Pr(Y)) \Rightarrow \Pr(Y) > \frac{1}{3}$

# Preliminary definitions

## Capacity

$\rho : 2^S \rightarrow [0, 1]$  such that:

- $\rho(\emptyset) = 0$  and  $\rho(S) = 1$
- $A \subseteq B \Rightarrow \rho(A) \leq \rho(B)$

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### Binary acts $\mathcal{B}$

$f \in \mathcal{A}$  st there exists  $E \in 2^S$ ,  $x, y \in Y$ :

- $f(s) = x, \forall s \in A$
- $f(s) = y, \forall s \in A^c$
- denoted:  $xAy$

# Biseparable preference (Ghirardato & Marinacci, 2001)

## c-linear biseparable preferences

The preference relation  $\succsim$  is c-linear biseparable iff there exist a function  $V : \mathcal{A} \rightarrow \mathbb{R}$  and a capacity  $\rho$  on  $2^S$  such that:

- $\forall x \succsim y$ , letting  $u(x) = V(x)$ ,

$$V(xAy) = \rho(A)u(x) + (1 - \rho(A))u(y)$$

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- $\forall f \in \mathcal{B}, x \in Y, \alpha \in [0, 1]$ ,

$$V(\alpha f + (1 - \alpha)x) = \alpha V(f) + (1 - \alpha)V(x)$$

Moreover  $\rho$  is unique and  $V$  is unique up to an increasing affine transformation.



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## Uncertainty aversion

A c-linear bisep. preference is *uncertainty neutral wrt event E* iff  $\rho(E) = 1 - \rho(E^c)$ .

It is *uncertainty neutral* if it is uncertainty neutral wrt all events.

# Biseparable preference (Ghirardato & Marinacci, 2001)

## Examples

- Subjective Expected Utility
- Choquet Expected Utility (Schmeidler, 1986)
- Maxmin Expected Utility (Gilboa & Schmeidler, 1989)
- $\alpha$ -Maxmin Expected Utility (Jaffray, 1989)

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## GTV 08

Generalization of c-linear biseparable preferences: rank dependent additive preferences allow for state dependence.

## Aggregation (Gajdos, Tallon, Vergnaud, 2008)

## Theorem

Assume that  $\succsim_i$  are c-linear biseparable preferences, represented by functions  $V_i$  with capacities  $\rho_i$  and that (IP) is satisfied. Then (WP) holds iff there exist unique  $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}_+^n \setminus \{0_n\}$ ,  $\mu \in \mathbb{R}$  such that for  $f \in \mathcal{B}$ :

$$V(f) = \sum_{i \in N'} \lambda_i V_i(f) + \mu.$$

Moreover,  $\lambda_i \lambda_j \neq 0$  iff  $i$  and  $j$  are uncertainty neutral

# Aggregation (Gajdos, Tallon, Vergnaud, 2008)

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## Example

- Non-dictatorial aggregation of Maxmin Expected Utility maximizers (or CEU) is impossible if individuals are uncertainty averse
- True **even** if they have the same "beliefs"

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Usual arguments show that aggregation must be linear



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- Again use (IP), assuming  $\exists \lambda_j > 0, \lambda_k > 0: \exists x, y$  st  $x \succ_j y, y \succ_k x$  and  $x \sim_0 y$   
 $V_0(xEy) - V_0(yEx) = 0$  (direct computation)

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$$V_0(xEy) - V_0(yEx) = \sum_i \lambda_i (V_i(xEy) - V_i(yEx))$$

$$\text{Given } \rho_i(E) = 1 - \rho_i(E^c), \text{ leads } \rho_0(E) = 1 - \rho_0(E^c)$$

**The End?**

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## Proof of Lemma 1

By (IP) and vNM there exist  $(p_k, q_k)$ ,  $k \in N'$  such that:

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Let  $p^* = \sum_i \frac{1}{n} p_i$  and  $p_* = \sum_i \frac{1}{n} q_i$ .

$$\begin{aligned} u_k(p^*) &= \sum_i \frac{1}{n} u_k(p_i) \\ &= \frac{1}{n} u_k(p_k) + \sum_{i \neq k} \frac{1}{n} u_k(p_i) \\ &> \frac{1}{n} u_k(q_k) + \sum_{i \neq k} \frac{1}{n} u_k(q_i) = u_k(p_*) \end{aligned}$$

## Proof of Lemma 2

### Definitions

- $R = \{z \in \mathbb{R}^{n+1} \mid z_0 \leq 0, z_i > 0 \forall i \in N'\}$
- $K = (f_0, f_1, \dots, f_n)(X) = F(X)$  convex
- $K^- = \{z' - z'' \mid (z', z'') \in K^2\}$  convex and symmetric wrt 0



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### Separation argument

- (WP)  $\Leftrightarrow R \cap K^- = \emptyset \Leftrightarrow R \cap \text{Vect}(K^-) = \emptyset$  ( $K^-$  conv and sym)

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- (WP)  $\Leftrightarrow R \cap K^- = \emptyset \Leftrightarrow R \cap \text{Vect}(K^-) = \emptyset$  ( $K^-$  conv and sym)
- $\tilde{R} = \text{cl}(R) + \sum_{i \in N} e_i \subset R$

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- $K^- = \{z' - z'' \mid (z', z'') \in K^2\}$  convex and symmetric wrt 0

### Separation argument

- (WP)  $\Leftrightarrow R \cap K^- = \emptyset \Leftrightarrow R \cap \text{Vect}(K^-) = \emptyset$  ( $K^-$  conv and sym)
- $\tilde{R} = \text{cl}(R) + \sum_{i \in N} e_i \subset R$
- Separation of closed disjoint non-empty polyhedral sets:  
 $\exists \varphi = (\varphi_0, \varphi_1, \dots, \varphi_n)$  st  $\forall k \in \text{Vect}(K^-), z \in \text{cl}(R)$ :  

$$\langle \varphi, z + \sum_{i \in N'} e_i \rangle > \langle \varphi, k \rangle$$

## Proof of Lemma 2

### Definitions

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## Proof of Lemma 2

Sign of  $\varphi_i$ ,  $i \in N'$

- $\gamma e_j + \sum_{i \in N'} e_i \in R, \forall \gamma > 0$
- $\langle \varphi, \gamma e_j + \sum_{i \in N'} e_i \rangle > 0$
- $\varphi_j(1 + \gamma) + \sum_{i \in N' \setminus \{j\}} \varphi_i > 0, \forall \gamma > 0$
- Thus  $\varphi \neq 0$ . Let  $\gamma \rightarrow \infty : \varphi_j \geq 0$

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### Sign of $\varphi_0$

- $[(MA) \text{ and } (WP)] \Rightarrow$  there exist  $(\theta_0, \theta_1, \dots, \theta_n) \in K^-$  s.t.  $\theta_i > 0$  for all  $i$
- $\varphi_0 \theta_0 = \sum_{i \in N'} -\varphi_i \theta_i$
- Thus  $\varphi_0 < 0$



## Proof of Lemma 3

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- $u_0(p_k) - u_0(q_k) = \lambda_k (u_k(p_k) - u_k(q_k))$
- Thus  $\lambda_k$  unique (true for all  $k \in N'$ )
- Thus  $\mu$  unique

## Diamond's critics

$p$	$Pr(\theta_1) = \frac{1}{2}$	$Pr(\theta_2) = \frac{1}{2}$
$u_a$	1	0
$u_b$	0	1

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The Independence assumption is unacceptable for the social preferences

# The identification problem

Sen (1986), Weymark (1991)

- Let  $\tilde{u}_i = \alpha_i u_i$
- $\tilde{u}_i$  is still a vNM representation of  $\succsim_i$
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Weights are meaningless from a normative point of view