On Lorenz Preorders and Opportunity Inequality in Finite Environments

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Part I

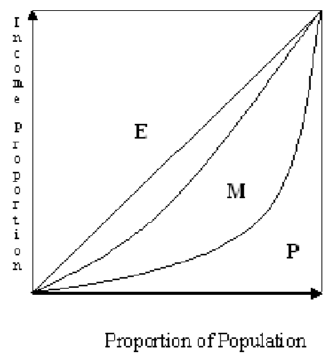
Introduction

It is well-known that early in the last century, economists became interested in evaluating inequality of incomes or of wealth. Thus, in order to measure how much desirable is a given distribution with respect to another one in terms of equality, it became important to determine on which basis one (income) distribution could be compared with another as ‘more even’.

- The first statement of this kind was made precise by Max Otto Lorenz (1876-1959), an applied statistician, who was Director of the Bureau of Statistics in the United States and who has developed in 1905 what is now well-known as the *Lorenz curve.*

- A second independent origin of Lorenz curve is in mathematical analysis and can be dated back to the work of Muirhead (1903) concerning a generalization of the arithmetic-geometric mean inequality.

Notation 1 *Consider a population of n individuals, whose income could be represented by a positive natural number. Order the individuals from the poorest to the richest to obtain a vector distribution $x_1, ..., x_n$, where the generic x_i denotes the wealth of individual i, $i = 1, ..., n$. Then, plot the points $\left(\frac{k}{n}, \frac{S_k}{S_n}\right)$, $k = 0, ..., n$, where $S_0 = 0$ and $S_k = \sum_{i=1}^{k} x_i$ is the total income of the poorest k individuals in the population. Hence, joining the points by line segments to obtain a curve connecting the origin with the point $(1, 1)$, we get a curve as M in the picture below, that is convex and lies under the straight line E representing the distribution where all people get the same quantity of income. The more closer a curve is to E, the more even is the distribution of income.*



LORENZ curves

Notation 2

Notation 3 *If $x_1, ..., x_n$ denote the incomes of individuals in the distribution of total income I that induce curve M and analogously $y_1, ..., y_n$ induce curve P, then according to the idea of Lorenz, $(x_1, ..., x_n)$ represents a 'more nearly equal' distribution of I than does $(y_1, ..., y_n)$ if and only if*

$$\sum_{i=1}^{k} x_i \geqslant \sum_{i=1}^{k} y_i, \qquad i = 1, ..., n-1 \tag{1}$$

$$\textit{and of course } \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$$

- Muirhead (1903) was the first who identified the relation 1, which is modernly denoted to as $\mathbf{x} \prec \mathbf{y}$, read as '$\mathbf{x}$ is majorized by $\mathbf{y}$' and represent a preorder on the space of all vector distributions in $\mathbb{R}^n$. Moreover, Muirhead proved that if the components of $\mathbf{x}$ and $\mathbf{y}$ are nonnegative integers, then condition 1 is equivalent to the fact that distribution $\mathbf{x}$ can be derived from distribution $\boldsymbol{y}$ by a finite number of transfers of income which take place from the richer to the poorer individual and that must not be so large enough to reverse the relative ranking of the two people involved in the transfer. This normative principle, now called Pigou-Dalton,[1] was actually first discussed by Muirhead (1903) and can be summarized as follows:

Definition 1 *Let $\mathbf{y} = (y_1, ..., y_n) \in \mathbb{R}_+^N$, then if y_l is the income of individual l, $l = 1, ..., n$, $y_i < y_l$ and an amount δ of income is transferred from individual l to i, than income inequality is diminished provided $\delta \leq \frac{(y_l - y_i)}{2}$.*

- A sequence of Pigou-Dalton transfers produces an 'averaging' over a distribution that is tantamount to pre-multiply a column vector $\mathbf{y}$ by a doubly stochastic matrix[2] B in order to obtain a smoother distribution $\mathbf{x}$, namely $\mathbf{x} = B\mathbf{y}$. In his seminal work on Hadamard's determinant inequality, Schur (1923) proved that if $\mathbf{x} = B\mathbf{y}$ for some doubly stochastic matrix B, then $\sum_{i=1}^{n} \varphi(x_i) \leq \sum_{i=1}^{n} \varphi(y_i)$ for all continuous concave function $\varphi : I \subset \mathbb{R} \to \mathbb{R}$.

[1] From Pigou that hinted at it on pg. 24 of his book "Wealth and Welfare" on (1912) and from Dalton, who clearly described the notion of transfer in an income distribution context in a seminal work in the Economic Journal on 1920.

[2] Remind that a $n \times n$ bistochastic matrix B is a nonnegative square matrix where all row and column sums are equal to 1.

Later, all the main elementary properties described above of the Lorenz preorder were summarized by the following classic result due to HLP (1934) (see also Marshall and Olkin(1979)), namely:

Theorem 1 *For any* $\mathbf{x} = (x_1, ..., x_n)$*,* $\mathbf{y} = (y_1, ..., y_n) \in \mathbb{R}^N$*, the following conditions are equivalent:*

(i) $\mathbf{y} \succcurlyeq^M \mathbf{x}$*, i.e.*

$$\sum_{h=1}^{k}(\mathbf{y}\downarrow)_h \geq \sum_{h=1}^{k}(\mathbf{x}\downarrow)_h, \quad h = 1, ..., n-1, \text{ and } \sum_{h=1}^{n}(\mathbf{y}\downarrow)_h = \sum_{h=1}^{n}(\mathbf{x}\downarrow)_h,$$

where, for any $\mathbf{u} = (u_1, ..., u_n)$*,* $\mathbf{u}\downarrow = (u_{\sigma(1)}, ..., u_{\sigma(n)})$*, with* $\sigma : N \longrightarrow N$ *a permutation such that* $\sigma(i) \geq \sigma(j)$*, entails* $u_i \leq u_j$*;*

(ii) $\mathbf{x}$ *can be derived from* $\mathbf{y}$ *through a finite sequence of transformations* $\mathbf{z}' = f(\mathbf{z})$ *of the following type:*

$$\begin{aligned} z_i' &= z_i + \delta \\ z_j' &= z_j - \delta \quad \text{with } j \leq i \\ z_k' &= z_k, \quad k \neq i, j \text{ and } \delta > 0 \end{aligned}$$

provided $\delta \leq (z_j - z_i)/2$*;*

(iii) $f(\mathbf{y}) \geq f(\mathbf{x})$ *holds for any* $f : \mathbb{A} \subset \mathbb{R}^N \rightarrow \mathbb{R}$ *of the following form: for each* $\mathbf{z} \in \mathbb{A}$*,* $f(\mathbf{z}) = \sum_{i=1}^{n} \varphi(z_i)$ *where* $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ *is a continuous convex function.*[3]

[3] Functions $f(.) = \sum \varphi(.)$ as defined above are indeed Schur convex, i.e. such that $f(\mathbf{y}) \geq f(\mathbf{x})$ whenever $\mathbf{y} \succcurlyeq^M \mathbf{x}$.

- Income is not the most appropriate space for distributive judgements:

  (α) It can be persuasively argued that on many occasions *opportunities* rather than income levels are the proper targets of redistributive policies. Indeed, since individuals have different abilities to convert their resources into well-being, it is widely recognized that income may be utterly inadequate as an indicator of the opportunity for welfare they enjoy. (see Rawls (1971), Sen (1985, 2003), Roemer (1996).

  (β) People in fact differ in many respects beside income and we must consider several individual characteristics/criteria if we want to assess their actual opportunities.

Opportunities take many forms. For example, an individual's opportunities are enhanced with greater wealth, grater access to education, and the removal of barriers based on race, gender, or religion on the choice of one's career.

- Unfortunately, it is not clear *if* and *how* those (personal) characteristics/criteria might be aggregated. In other words, the analysis of opportunity in terms of inequality is complicated by the fact that an individual's opportunities are described by a set rather than by a scalar, as in the case with income or wealth inequality.

So a **new problem** rapidly arises:

0.0.1 How to rank opportunity sets

Problem 1 *How to compare (and rank) sets (of opportunities)?*

Solution 1 (Preference-for-flexibility) *Assume that there exists a nonempty set of alternatives X and an individual preference ordering R on X which, unless stated otherwise, is assumed to be linear. A relation $\succeq$ on the set of all nonempty and finite subsets $\mathcal{P}(X)$ of X (which are interpreted as opportunity sets or menus from which an agent can make a choice) is to be established.*

Example 1 (Kreps (Econ. 1979)) *The indirect-utility ranking $\succeq_U$ of opportunity sets is defined by letting, for all $A;B \in \mathcal{P}(X)$*

$$A \succeq_U B \text{ if and only if } \max(A)\, R \max(B)$$

That is, only the best elements according to R in the sets to be compared matter in establishing an ordering on X. Kreps (Econom.1979) provides a characterization of the indirect-utility criterion in a model where R is not fixed. The axiom used in this characterization is the following extension-robustness condition. It requires that adding a set B that is at most as good as a given set A to A leads to a set that is indifferent to A itself.

Axiom 1 *For all $A;B \in \mathcal{P}(X)$,*

$$A \succeq B \to A \sim A \cup B.$$

Proposition 1 *An ordering on X satisfies Extension Robustness if and only if there exists an ordering R on X such that is the indirect-utility ranking for the ordering R.*[4]

The indirect-utility criterion is based on the position that the quality of the final choice of the agent is all that matters, and the only reason other characteristics of an opportunity set might be of interest is that they may have instrumental value in achieving as high a level of well-being as possible.

Claim 1 (i) *The way alternatives are formulated in economic models is often very restrictive, and they may not capture everything of value to an agent. In this case, utility is not an indicator of overall well-being but, rather, a measure of one aspect of well-being.*

[4] Note that the extension-robustness axiom does not make any reference to an underlying ordering.

(ii) The ranking of opportunity sets should arguably only take into account the 'size' of the relevant set, without making any use of information about individual preferences which may be highly unreliable, costly to acquire, *or both.*[5]

Solution 2 (Freedom of choice) *We might want to rank opportunity sets in terms of the freedom of choice they offer. The conceptual basis of many of the formal models on the ranking of opportunity sets in terms of freedom is to be found in the belief that freedom of choice has a value that is independent of the amount of utility that may be generated by such freedom. Then one can think of the volume of options figuring in the opportunity set as to be relevant. When the number of options is finite, the simplest way of assessing the volume or quantity of options available to the agent is to count how many options there are in the (opportunity) set.*

Example 2 (Pattanaik and Xu (1990)) *They follow an axiomatic approach to the problem, and use three axioms to characterize a rule for ranking finite opportunity sets on the basis of their cardinalities. This cardinality-based ordering $\succeq_C$ X is defined by letting, for all A;B $\in \mathcal{P}(X)$*

$$A \succeq_C B \text{ if and only if } |A| \geq |B| .$$

***Axiom 2 (Indifference Between No-Choice Situations** (NC)**)** *For all x, $y \in X$,*

$$\{x\} \sim \{y\} .$$

***Axiom 3 (Simple Expansion Monotonicity** (M)**)** *For all distinct x, $y \in X$,*

$$\{x, y\} \succ \{y\} .$$

***Axiom 4 (Strong Independence** (IND)**)** *For all A;B $\in \mathcal{P}(X)$ and $x \in X \setminus (A \cup B)$,*

$$A \succeq B \text{ if and only if } A \cup \{x\} \succeq B \cup \{x\} .$$

Theorem 2 *Suppose $\succeq$ is a complete preorder on $\mathcal{P}(X)$. Then, $\succeq$ satisfies NC, M, IND if and only if $\succeq = \succeq_C$.*

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[5] Such a 'freedom of choice'-based ranking might arguably also be relevant in order to define a suitable "capability"-ordering of opportunity sets.

It turns out that the (total) cardinality-based preorder $\succcurlyeq_C$ does support a Lorenz-like preorder of opportunity distributions over a finite opportunity set, as shown by Ok and Kranich (1998).

Notation 4 *Let X denote the* finite *set of alternatives/opportunities, $N = \{1, ..., n\}$ the population of agents, and $\mathcal{P}(X)$ the power set of X, i.e. the set of its opportunity sets, with $\#X \geq 3$, in order to avoid trivial qualifications. We are interested in those opportunity rankings $(\mathcal{P}(X), \succcurlyeq)$ that arise whenever all the alternatives are "never bad".*

Theorem 3 (Ok and Kranich (1998)) *If $(\mathcal{P}(X), \cup, \succcurlyeq_{\#})$ then for any pair of opportunity distributions $\mathbf{A}, \mathbf{B} \in (\mathcal{P}(X))^N$, $\mathbf{A} \prec \mathbf{B}$ iff $\mathbf{A}$ is reachable from $\mathbf{B}$ through a finite sequence of (suitably defined) Pigou-Dalton transfers or iff $f(\mathbf{A}) \leqslant f(\mathbf{B})$ for any (suitably defined) Schur-concave function f.*

Moreover, the cardinality-based preorder is the sole **Strict Set-Inclusion Monotonic total preorder** which supports such a Lorenz-like preorder of opportunity distributions, namely:

Theorem 4 (Ok (1997)) *Let $(\mathcal{P}(X), \cup, \succcurlyeq)$ be such that $(\mathcal{P}(X), \succcurlyeq)$ is a Strict Set-Inclusion Monotonic totally preordered set,*[6] *and for any pair of opportunity distributions $\mathbf{A}, \mathbf{B} \in (\mathcal{P}(X))^N$, $\mathbf{A} \prec \mathbf{B}$ iff $\mathbf{A}$ is reachable from $\mathbf{B}$ through a finite sequence of (suitably defined) Pigou-Dalton transfers or iff $f(\mathbf{A}) \leqslant f(\mathbf{B})$ for any (suitably defined) Schur-concave function f. Then $\succcurlyeq = \succcurlyeq_{\#}$.*

[6] Or, alternatively, a totally preordered set satisfying another property called 'contraction consistency' (see Ok (1997) for a formal definition and a discussion of such a property).

Critics to cardinality total preorder of opportunity sets

(i) However, the cardinality-based preorder is typically regarded as a trivial and uninteresting ranking of opportunity sets.

(ii) Indeed, rejection of the cardinality ranking is quite obvious if some relevant evaluations of (or preferences on) opportunities are available. But the cardinality-based preorder is also typically rejected as a ranking of opportunity sets in terms of *freedom of choice*, namely when reliable and detailed preferential information is not available or is deemed to be not relevant.

(iii) We concur, but one should then ask: why is that so? The answer to the foregoing question is not entirely clear, but one of the main drawbacks of the cardinality-based preorder is arguably its failure to accommodate even a minimal diversity of judgments concerning the relevance of distinct opportunities.

Part II

Ranking opportunity profiles

0.0.2 Other two extra complications in comparing sets of opportunities

1. While income is a private good hence both excludable and rivalrous, opportunities may be non-rivalrous and possibly non-excludable according to the nature of the goods they are attached to. Thus, opportunities may be conceivably of a *private*, *public* or pure-public type.

2. It can be plausibly maintained that opportunities as opposed to the income levels are inherently *multidimensional* objects.

Claim 2 *As a consequence of so-many complications, it was only with the seminal work of Kranich (Jet, 1996) that the question of how to rank different distributions of opportunities in terms of inequality they exhibit was first addressed.*

0.0.3 Literature

The beginning: Laurence Kranich and Efe Ok

(i) **Kranich** (Jet, 1996):

Kranich is concerned with the problem of establishing an ordering in terms of equality on the set of possible distributions of finite opportunity sets among the individuals in a given society. He starts with a two-individual society and seeks to characterize a specific equality ordering on $\mathcal{P}(X) \times \mathcal{P}(X)$, Then, each element $(A_1, A_2) \in \mathcal{P}(X)^2$ represents a distribution of opportunity sets, where individual 1 has the opportunity set A_1 and individual 2 has the opportunity set A_2. For a two individual society, Kranich imposes the following axioms on an equality ordering $\succeq_e$ on $\mathcal{P}(X)^2$.

Axiom 5 (Anonimity) *For all* $(A_1, A_2) \in \mathcal{P}(X)^2$*,*

$$(A_1, A_2) \sim_e (A_2, A_1) .$$

Axiom 6 (Monotonicity of Equality) *For all* $A_1, A_2, A_3 \in \mathcal{P}(X)$ *such that* $A_1 \subseteq A_2 \subset A_3$*,*

$$(A_1, A_2) \succ_e (A_1, A_3) .$$

Axiom 7 (Independence of Common Expansions) *For all* $A_1, A_2, A_3 \in \mathcal{P}(X)$ *such that* $A_3 \cap (A_1 \cup A_2) = \varnothing$

$$(A_1 \cup A_3, A_2 \cup A_3) \sim_e (A_1, A_2) .$$

Axiom 8 (Assimilation) *For all* $(A_1, A_2) \in \mathcal{P}(X)^2$*, for all* $a \in A_1$*,* $b \in A_2$*,* $c \in (X \backslash [(A_1 \backslash \{a\}) \cup (A_2 \backslash \{b\})])$*,*

$$((A_1 \backslash \{a\}) \cup \{c\}, (A_2 \backslash \{b\}) \cup \{c\}) \succeq_e (A_1, A_2) .$$

The axiom of independence of common expansions is not so-plausible. It requires that if we discard some options common to the opportunity sets of both individuals, then the degree of equality remains unchanged. Similarly, assimilation can be challenged as well. This axiom requires that if we replace an arbitrary element $a \in A_1$ and an arbitrary element $b \in A_2$ with an option c that is neither in $A_1 \backslash \{a\}$ nor in $A_2 \backslash \{b\}$, then the degree of equality cannot decrease.

Kranich shows that the only equality ordering $\succeq_e$on $\mathcal{P}(X)^2$ satisfying the above four axioms is the cardinality-difference ordering $\succeq_{CD}$defined as follows.

Definition 2 *For all* (A_1, A_2)*,* $\left(A_1^{'}, A_2^{'}\right) \in \mathcal{P}(X)^2$*,*

$$(A_1, A_2) \succeq_{CD} \left(A_1^{'}, A_2^{'}\right) \Longleftrightarrow ||A_1| - |A_2|| \geq \left|\left|A_1^{'}\right| - \left|A_2^{'}\right|\right|.$$

Kranich (Jet, 1996) relies on *differences* between *cardinalities* of opportunity sets in order to address the issue of ranking *profiles* of *non-rivalrous* opportunities in terms of inequality. In that connection, he provides a characterization of a class of indices of opportunity (in)equality that is strictly related to the class of generalized Gini inequality indices.

Herrero, Iturbe-Ormaetxe, and Nieto (Mass, 1998)

(ii) In contrast to Kranich (Jet, 1996), Herrero, Iturbe-Ormaetxe, and Nieto (Mass, 1998) focus on another aspect of one's intuition about the equality of opportunities (see also Bossert, Fleurbaey, and Van de gaer (Rev. Econ. Des., 1999) and Kranich (Jet, 1996)). For a two-person society, one can plausibly argue that the larger the number of options that are common between the opportunity sets of two individuals, the higher the degree of equality. More generally, for a society with any number of individuals, this argument extends to the view that the larger the number of options that are common to the opportunity sets of all individuals, the higher the degree of equality. In other words, Herrero, Iturbe-Ormaetxe, and Nieto (Mass, 1998) do rely on *total* preorders of opportunity sets, and provide characterizations of several egalitarian and utilitarian-like *total* preorders of opportunity profiles, including a few new criteria which emphasize the role of common opportunities (i.e. opportunities belonging to every component of the relevant profile). Similarly, Arlegi and Nieto (LGS1, 1999) offer axiomatizations of certain *total (in)equality* preorders of opportunity profiles which only depend on cardinality differences and/or the number of common opportunities.

Imposibility results

(iii) Ok and Kranich (SCW, 1998) also consider the issue of the equality of a distribution of opportunity sets. They focus on the case of a

two-individual society where, for each individual, the alternative opportunity sets are ranked on the basis of their cardinalities. In this framework, they prove an analogue of a basic theorem in the literature on the measurement of income inequality. They first introduce the notion of an equalizing transformation of a given pair of opportunity sets in their two-person society and also the notion of a Lorenz quasi-ordering on the set of pairs of opportunity sets. The main result of Ok and Kranich (SCW, 1998) shows that, in their assumed framework, one distribution of opportunity sets Lorenz dominates another distribution if and only if the rst distribution can be reached from the second by a nite sequence of equalizing transformations and if and only if every inequality-averse social welfare functional ranks the first distribution higher than the second. The Lorenz quasi-ordering that Ok and Kranich (SCW, 1998) generate in a two-individual society does not satisfy Kranich's (Jet, 1996) highly plausible axiom, monotonicity of equality. This, of course, is a problem similar to one that arises in the context of income distribution. Given this problem, Ok and Kranich (1998) explore various extensions, satisfying monotonicity of equality, of their Lorenz quasi-ordering.

Ok (Jet, 1997)

A general result due to Ok (Jet, 1997) has a pessimistic message regarding the possibility of measuring inequality in the distribution of opportunities. Ok formulates the counterpart of the fundmental concept of an equalizing transfer familiar in the literature on income distribution (see Dalton [39]), and he does this in a way that is more general than the corresponding formulation in Ok and Kranich (SCW, 1998). The formulation of the notion of an equalizing transfer here has to capture the intuition that a 'transfer' of opportunities from a person with 'more' opportunities to a person with 'fewer' opportunities increases equality, provided that such a transfer does not reverse the ranking of the two individuals in terms of the opportunities available to them. Therefore, any such formulation has to be based on some ranking of the opportunity sets that gives us an ordinal measure of the 'amount' of opportunities reflected in an opportunity set. Ok (Jet, 1997) introduces certain plausible formulations of an equalizing transfer with respect to opportunities, and shows that the only ranking of opportunity sets that can serve as a basis of the notion of an equalizing transfer, as formulated by him, must be the cardinality-based ranking. Since the requirements that Ok (Jet 1997) imposes on the concept of an equalizing transfer of opportunities are intuitively very appealing, his central results have a strong negative flavour, given the restrictive nature of the cardinality-based ranking of opportunity sets.

Part III

Filtral preorders and opportunity inequality

From this review of the relevant literature, it appears that the majorization preorder of opportunity profiles is currently confined to a relatively marginal role. It seems to us that the prevailing interpretation of Ok's theorem as mentioned above may partly explain such an attitude to downplay majorization. Indeed, the cardinality preorder of opportunity sets is commonly (and rightly) rejected as trivial. But then, if the cardinality preorder is the *only* one which supports an opportunity counterpart to the classic HLP theorem, it follows that the main result in Ok (Jet, 1997) is to be regarded as an 'impossibility theorem' of sorts on majorization-consistent rankings of opportunity profiles (see also Barberà et alii (2004) on this point).

- In our view, such an interpretation should be firmly resisted. It should be recalled that Ok's (1997) results only hold for *strictly inclusion-monotonic* (or 'contraction-consistent' inclusion-monotonic) rankings. We simply propose to relax those stringent requirements by focusing *on the entire class of inclusion-monotonic rankings.* Then, we show that in this broader environment it is after all *possible* to extend the celebrated HLP theorem to the measurement of opportunity inequality even starting from several preorders of opportunity sets which are different from the cardinality preorder.

- In that connection, we propose to rely on a class of minimal extensions of the set-inclusion partial order whose members we shall refer to as *set-inclusion filtral preorders* (henceforth SIFPs). A SIFP on a (finite nonempty) set X of basic alternatives/opportunities amounts to an elementary way to augment the set-inclusion partial order with a *minimum opportunity-threshold*: under the threshold, opportunity sets are indifferent to each other and to the null opportunity set, while over the threshold the set-inclusion partial order is simply replicated.

0.1 Setting

We are interested in a class of opportunity rankings that arise whenever:

1. all the alternatives are "never bad": indeed, they are typically "good" but not necessarily so;[7]

2. as a consequence of (1), the set-inclusion ordering is *weakly* respected but not strictly so, i.e. our rankings are set-inclusion monotonic (as opposed to strictly set-inclusion monotonic);

3. a minimum standard (threshold) is introduced such that any opportunity set which does not meet the corresponding standard is simply not acceptable i.e. is equivalent to the null set.

Thus, we focus on a preordered set $(\wp(X), \succcurlyeq)$ which extends the set-inclusion poset, namely $A \supseteq B$ entails $A \succcurlyeq B$ for all $A, B \in \wp(X)$.[8] (As usual, we denote by $\succ$ and $\sim$ the asymmetric and symmetric components of $\succcurlyeq$, respectively). In order to capture the notion of a threshold in this setting, we shall rely on the definition of an *order filter of a poset*. In fact, such an order filter collects all the elements of the poset which are greater than some member of a specified list of *noncomparable* elements, the so-called *generators*.

Definition 3 ((Principal) Order Filters of a Poset) *Let $(Y, \succcurlyeq)$ be a non-empty poset and $\mathcal{B}$ an* antichain *of $(Y, \succcurlyeq)$, namely $\mathcal{B} \subseteq Y$ and for any $b_i, b_j \in \mathcal{B}$ if $b_i \neq b_j$ then* not *$b_i \succ b_j$. An order filter of $(Y, \succcurlyeq)$ with basis $\mathcal{B}$ is a set $F = F(\mathcal{B}) \subseteq Y$ such that*

1. *$\mathcal{B} \subseteq F$ and*

2. *for any $A, B \in Y$, if $A \in F$ and $B \succ A$ then $B \in F$.*

Whenever, $\#\mathcal{B} = 1$, i.e. $\mathcal{B}$ is a singleton, F is a principal *order filter.*

[7] By considering alternatives which are not good in a strict sense, we allow for complementarity effects among opportunities.

[8] A preordered set or preposet is a pair $(Y, \succcurlyeq)$ such that Y is a set and $\succcurlyeq$ is a reflexive and transitive binary relation on Y. A poset is a preposet $(Y, \succcurlyeq)$ such that $\succcurlyeq$ is antisymmetric.

We shall use principal order filters of the set-inclusion poset $(\wp(X), \supseteq)$ to introduce a *filtral extension* of the latter. As explained previously, this amounts to enriching $(\wp(X), \supseteq)$ with a suitable threshold, which, in turn, corresponds to the requirement of a minimum level of freedom, i.e. an opportunity poverty line of sorts. To repeat, below the threshold the available amount of individual *freedom of choice* is deemed to be *not* acceptable.[9] All this can be embodied in the following:

Definition 4 (Set-Inclusion (Principal) Filtral Preorders (SIFPs)) *For any (principal) order filter F of $(\wp(X), \supseteq)$ the F-generated* set-inclusion (principal) filtral preorder (SIFP) *is the binary relation $\succcurlyeq_F$ on $\wp(X)$ defined as follows: for any $A, B \in \wp(X)$, $A \succcurlyeq_F B$ if and only if $A \supseteq B$ or $B \notin F$.*[10]

[9] Think e.g. of a citizen that has access to any newspaper she likes to read and enjoys freedom of speech but is deprived of voting rights.

[10] Notice that, under the extremal or degenerate cases $F = \wp(X)$ and $F = \varnothing$, $(\wp(X), \succcurlyeq_F)$ reduces to the set-inclusion order and the degenerate total preorder consisting of a single indifference class, respectively.

As mentioned in the Introduction, the main aim of the present paper is to propose a *SIFP*-based method of ranking profiles of opportunity sets in terms of opportunity inequality. In order to accomplish the foregoing task we have to introduce the following:

Definition 5 *Let F be a (principal) order filter of $(\wp(X), \supseteq)$ and $\succcurlyeq_F$ the (principal) SIFP induced by F. Then, the $\succcurlyeq_F$*-induced height function

$$h_{\succcurlyeq_F} : \wp(X) \to \mathbb{Z}_+$$

is defined as follows: for any $A \subseteq X$:

$$h_{\succcurlyeq_F}(A) = \max \left\{ \begin{array}{c} \#\mathcal{C} : \mathcal{C} \text{ is a } \succcurlyeq_F \text{-chain, such that} \\ A \in \mathcal{C} \text{ and } A \succ_F B \text{ for any } B \in \mathcal{C} \setminus \{A\} \end{array} \right\}.$$

In words, the height function assigns to each opportunity set A a non-negative number, namely the size of the longest strictly ascending chain having A as its maximum.[11]

- Let us now outline a description of our approach to the issue of inequality ranking of opportunity profiles. We start by adjoining a (principal filter-induced) threshold to the set inclusion ordering of opportunity sets, which, of course, provides a (principal) filtral opportunity preorder.

- Then, we consider the resulting heights for the opportunity profiles under consideration. Next apply the majorization preorder of Theorem of HLP above to the set of height profiles we obtain. Such a preorder induces in a natural way a preorder on opportunity profiles. This is an inequality ranking of opportunity profiles, which is a counterpart of the (*dual of*) Lorenz ranking of income distributions.

[11] Recall that a chain of a preordered set $(Y, \succcurlyeq)$ is a subset $Z \subseteq Y$ such that $(Z, \succcurlyeq)$ is a totally preordered set.

Thus the procedure we propose can be summarized as follows:

1. take a principal *SIFP* $(\wp(X), \succcurlyeq_F)$ on $(\wp(X), \supseteq)$;

2. consider the $\succcurlyeq_F$-induced height function $h_{\succcurlyeq_F}$;

3. use the majorization preorder on height profiles in order to define a generalized $\succcurlyeq_F$-induced majorization preorder $\succcurlyeq_F^M$ on the set $(\wp(X))^N$ of N-profiles of opportunity sets.

We denote the set of all admissible opportunity profiles for population N as $(\wp(X))^N$. With a slight abuse of notation, we denote by $\mathbf{A} = (A_i)_{i \in N}$ a generic opportunity profile. Hence, for any $i \in \{1, ..., N\}$, A_i represents the set of opportunities allotted to individual (or group) i according to $\mathbf{A}$. Therefore, we proceed to define an opportunity-profile-counterpart of the majorization preorder as defined in the Theorem of HLP (i):

Definition 6 *Let* $\mathbf{A}, \mathbf{B} \in (\wp(X))^N$ *be two opportunity profiles,* F *a principal order filter of* $(\wp(X), \supseteq)$, $\succcurlyeq_F$ *the corresponding set-inclusion (principal) filtral preorder (SIFP) on* $\wp(X)$, *and* $h_{\succcurlyeq_F}$ *the* $\succcurlyeq_F$*-induced height function on* $\wp(X)$. *Then* $\mathbf{A}$ *majorizes* $\mathbf{B}$, *denoted* $\mathbf{A} \succcurlyeq_F^M \mathbf{B}$, *if*

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i \tag{2}$$

and

$$\left(h_{\succcurlyeq_F}(A_1), ..., h_{\succcurlyeq_F}(A_n)\right) \succcurlyeq^M \left(h_{\succcurlyeq_F}(B_1), ..., h_{\succcurlyeq_F}(B_n)\right), \tag{3}$$

where $\succcurlyeq^M$ *denotes the majorization preorder as defined above under Theorem* $1.i$. *We denote by* $\succ_F^M$ *the asymmetric component of* $\succcurlyeq_F^M$.

Thus we focus on a quite general domain of opportunity profiles. One characteristic of $((\wp(X))^N, \succcurlyeq_F^M)$ is that it works by mapping the space of opportunity profiles into a set of integer points in $\mathbb{Z}_+^N$, i.e. the space of height vectors. This set will of course depend on the relevant principal order filter F and is therefore denoted as the (height) *span* of $\succcurlyeq_F$, written $H_{\succcurlyeq_F}$.

It should also be remarked here that, in general, height-based total extensions (HTE) of SIFPs, while being of course inclusion-monotonic total preorders, *do not satisfy* strictly inclusion monotonicity. Moreover, HTE of SIFPs do not satisfy Ok's 'contraction consistency' property. Hence, the fact that HTE of SIFPs do support an opportunity profile version of the HLP theorem is due precisely to their failure to satisfy the two restrictions considered by Ok (Jet, 1997)

We emphasize that the restriction to principal order filters we invoke does not come entirely for free. Indeed, as mentioned in the Introduction, opportunities are attached to resources which may be either rivalrous or non-rivalrous, and excludable or not. However, if the threshold amounts to a *unique* minimal opportunity set (according to the definition of a principal order filter) *and* such a minimal opportunity set includes *one or more rivalrous opportunities* (e.g. a *private* opportunity), then within our framework for each possible allocation of opportunities there can be at most *one* population unit that stands over or above the threshold. It is easily checked that, under this case, the majorization preorder reduces to its symmetric component. Hence, our main result, which provides two equivalent representations of the *asymmetric* component of the majorization preorder, reduces to a trivially true statement. On the other hand, and independently of the number of filter's generators, if *every* opportunity in any minimal opportunity set happens to be *non-excludable* (e.g. a pure public opportunity for the relevant population), then, by construction, all the population units stand over the opportunity threshold, which has therefore no tangible effect within our model. Thus, a sensible interpretation of our model requires that *all* opportunities in the unique generator of the relevant principal order filter be *non-rivalrous,* and at least *some* of them be *excludable.* The most straightforward way to ensure that, without any further ado, is assuming that *all opportunities in the basic set X are both non-rivalrous and excludable,* that is in fact the interpretation we suggest, and henceforth ask the reader to subscribe to.

0.2 A long digression

We are considering those Lorenz-style preorders of opportunity distributions which satisfy a counterpart of the foregoing HLP theorem.

- However, the Lorenz-based comparisons of univariate distributions are allowed by the *total* ordering induced by the perfect comparability of the individual incomes. On the contrary, individual endowments, namely multivariate distributions of personal goods/alternatives (hence opportunities), typically admit only *partial* non-controversial orderings.
- Therefore, a Lorenz preorder of opportunity distributions requires the preliminary definition of a *total* preorder on opportunity sets, an apparently controversial task.

As a matter of fact, the problem of building up a Lorenz-like preorder, starting from a partial (pre)ordering in a finite setting, has not received yet in the literature the attention it deserves. The few exceptions include some works such as Hwang (1979), Lih (1982), Hwang and Rothblum (1993), which focus on Lorenz preorder when the set of population units is endowed with a **fixed partial order**.

0.2.1 Majorization on partial orders

▲ Hwang (Proc. Amer. Mathe. Soc, 1979) extends the classical concept of Lorenz (or dually majorization) preorder on a set of distributions to the case where the set of coordinates or equivalently of population units is partially ordered. His results rely, quite unexpectedly, on a classical theorem of Shapley on the existence of the core for every convex game and parallel the mentioned result of Muirhead on the equivalence between the Lorenz order and a Pigou-Dalton finite sequence of transfers.

■ Lih (Siam J. of Agebr. Discr. Meth., 1982) also extends the concept of majorization as introduced above (see 1) to the case of real-valued functions defined on a finite partially ordered set. More precisely, Lih defines the majorization preorder as follows: let $(P, \leq)$ denote a finite poset and Φ the set of all real-valued functions on $(P, \leq)$, if $\alpha, \beta \in \Phi$ then α majorizes β if, for any order filter U of $(P, \leq)$, $\alpha(U) \geq \beta(U)$ and $\alpha(P) = \beta(P)$ where, for any $\gamma \in \Phi$, $\gamma(U) = \sum\{\gamma(x) : x \in U\}$. In such a setting, he replicates the classical result of HLP reviewed above.

♠ Hwang and Rothblum (Mathem. Oper. Res, 1993) provide a further extension of majorization and Schur convexity with respect to partial orders over the coordinates of an Euclidean space. They introduce the notion of 'pairwise connectedness' with respect to posets, which is actually a generalization of the Pigou-Dalton criterion of transfers, in order to achieve a characterization of Schur convexity (namely condition (*iii*) in Theorem of HLP above), with respect to partial orders for the case when every Schur convex function is neither necessarily symmetric (as in Lih (1982) and in the original work of Schur (1923)) nor asymmetric (as in Hwang (1979)). They also provide necessary and sufficient conditions for Schur convexity which rely on two-coordinate local properties of functions. That result implies that conclusions about local behavior of functions can be drawn without being forced to check every pair of coordinates on the (symmetric) domain of the function. Hence, their characterization of majorization via Schur convexity applies to a wider class of functions than those which are continuously differentiable. In particular, Hwang and Rothblum extend the original Schur-(Ostrowski) theorem (see Marshall and Olkin (1979) chapter 3)). Such a generalized approach to majorization with a partial order of population units or dimensions might conceivably be extended to a finite environment.[12]

[12] Hwang and Rothblum (1994) provide an example concerning *discrete* coherent systems composed of series modules.

However, as mentioned above, the basic source of difficulty, and controversy, in the economic literature, is uncertainty about the underlying 'basic' preorder on endowments or opportunity sets in a multidimensional setting.

0.3 Inequality ranking of opportunity profiles

Now, in order to replicate the basic findings of the literature on income inequality in the opportunity-profile setting, we must provide a suitable reformulation of the Pigou-Dalton transfer principle. Therefore, we first address the problem of defining a suitable notion of *transfer* with respect to height-extensions of SIFPs. Following Ok (Jet, 1997), we define a *weakly equalizing, or Daltonian, transfer operator.* Those notions are made precise by the following definitions.

Definition 7 *A transfer operator on* $(\wp(X))^N$ *is a nonempty correspondence* $\Im : (\wp(X))^N \rightrightarrows (\wp(X))^N$ *such that*

$$\forall (\mathbf{A}, \mathbf{B}) \in (\wp(X))^N \times \Im\left((\wp(X))^N\right) : \left[\bigcup_i A_i = \bigcup_i B_i\right]$$

Thus, a transfer operator is a transformation which leaves the set of total opportunities in $\mathbf{A}$ and $\mathbf{B}$ unchanged. Next, we define a notion of simple or minimal bilateral transfer:

Definition 8 *Let* $\mathbf{A}, \mathbf{B} \in (\wp(X))^N$ *be two opportunity profiles,* F *a principal order filter of* $(\wp(X), \supseteq)$ *and* $i, j \in N$ *such that* $A_j \succ_F A_i$*,* $x \in A_j \setminus A_i$*,*

$$\begin{aligned} B_j &= A_j \setminus \{x\} \\ B_i &= A_i \cup \{x\} \\ B_k &= A_k \quad k \neq i, j \end{aligned} \tag{4}$$

Then $\mathbf{B}$ *is said to arise from* $\mathbf{A}$ *through a* simple *(i.e. bilateral and minimal) transfer (from* j *to* i*). A transfer operator* $\Im$ *on* $(\wp(X))^N$ *shall be said* simple *if for any* $\mathbf{A}, \mathbf{B}$ *such that* $\mathbf{B} \in \Im(\mathbf{A})$*,* $\mathbf{B}$ *arises from* $\mathbf{A}$ *through a simple transfer.*[13]

[13] Notice that our definition of simple transfer operator amounts to a special case of the (n-person extension of) simple transfer operator as defined in Ok [**?**]. A possible alternative definition of simple transfer operator could be provided in terms of heights as follows: For any pair $(\mathbf{A}, \mathbf{B}) \in (\wp(X))^N \times \Im\left((\wp(X))^N\right)$: $[\max\{(h_{\succcurlyeq_F}(A_i) - h_{\succcurlyeq_F}(A_j)), (h_{\succcurlyeq_F}(A_j) - h_{\succcurlyeq_F}(A_i)), (h_{\succcurlyeq_F}(B_i) - h_{\succcurlyeq_F}(B_j)), (h_{\succcurlyeq_F}(B_j) - h_{\succcurlyeq_F}(B_i))\} \leq 1]$.
However, in this paper, we stick to 4 since it seems to be both more intuitive and more faithful to the notion of transfer in the Pigou-Dalton tradition.

In such a case, opportunity profile **B** is obtained from the opportunity profile **A** simply by transferring an item from a richer (in terms of opportunities) group of individuals to a poorer one. It should be noticed that our definition implicitly requires at least some basic opportunities to be excludable.

By analogy with the Pigou-Dalton principle, we also require that transfers of opportunities be not large enough to reverse the relative positions of the donor and recipient. This is the rationale of the next definitions, namely:

Definition 9 *A transfer operator* $\Im$ *is said to be:*

i) weakly rank-monotonic *w.r.t* $\succcurlyeq_F$ *if and only if it does not cause height-reversals i.e. for any* $\mathbf{A}, \mathbf{B} \in (\wp(X))^N$ *and any* $i, j \in N$,

$$\text{if } \mathbf{B} \in \Im(\mathbf{A}), B_i \neq A_i, B_j \neq A_j \text{ and } h_{\succcurlyeq_F}(A_j) \geq h_{\succcurlyeq_F}(A_i)$$

$$\text{then } h_{\succcurlyeq_F}(B_j) \geq h_{\succcurlyeq_F}(B_i);$$

ii) weakly progressive *w.r.t.* $\succcurlyeq_F$ *if and only if for any* $\mathbf{A}, \mathbf{B} \in (\wp(X))^N$:

$$\text{if } \mathbf{B} \in \Im(\mathbf{A}),\ B_i \supset A_i \text{ and } A_j \supset B_j$$

$$\text{then } h_{\succcurlyeq_F}(A_j) \geq h_{\succcurlyeq_F}(A_i).$$

iii) weakly-equalizing, *or* Daltonian, *w.r.t.* $\succcurlyeq_F$ *if it is simple, weakly rank-monotonic w.r.t.* $\succcurlyeq_F$ *and weakly progressive w.r.t.* $\succcurlyeq_F$.[14]

Another important fragment of the theory of income inequality, that we would like to know if it also holds in the present framework, concerns a Lemma due to Muirhead (see Marshall and Olkin (1979, chapter 1), that says that it is possible to obtain a income distribution $\mathbf{q}$ from $\mathbf{p}$ throughout a finite sequence of simple transfers of money, which minimally alter the initial distribution.

Let $\Im$ be a transfer operator, then, for any positive integer t and $\mathbf{A} \in (\wp(X))^N$ we define inductively $\Im^{(t)}(\mathbf{A}) = \Im(\Im^{(t-1)}(\mathbf{A}))$.

[14] Note that the *connectedness* property of definition 3.3 [(iii)] in Ok (1997), which requires the possibility to reach an egalitarian opportunity distribution by a finite sequence of equalizing transfers, cannot be satisfied by a simple transfer operator as defined in our setting, when starting from a non-egalitarian profile. We might achieve connectedness by substituting an opportunity *multiset* over X, namely a function $\mathbb{X} : X \to \mathbb{Z}_+$, for the original opportunity set X. We leave an exploration of this extension as a topic for further research.

Proposition 2 *Let F be a principal order filter of $(\wp(X), \supseteq)$, and $\mathbf{A}, \mathbf{B} \in (\wp(X))^N$ two opportunity profiles such that $\{h_{\succcurlyeq_F}(\mathbf{A}), h_{\succcurlyeq_F}(\mathbf{B})\} \subseteq H^+_{\succcurlyeq_F}$. Then, the following statements are equivalent:*

1. $\mathbf{A} \succ^M_F \mathbf{B}$;

2. *There exists a* $\succcurlyeq_F$-Daltonian transfer operator $\Im$ *and a positive integer* k *such that* $\mathbf{B} \in \Im^{(k)}(\mathbf{A})$.[15]

Notice that the foregoing Proposition is strictly related, but does not reduce, to the classic result on integer majorization due to Muirhead (see e.g. Marshall and Olkin (1979)). Here, transfers involve 'objects' or 'opportunities' and are only indirectly reflected by numbers. Hence, a remote control problem of sorts obtains concerning transfers: one has to double check that numbers always change in the 'right' direction by avoiding losses in the sum total of height-values due to equalizing opportunity-transfers of the wrong (i.e. height-destroying) type. Of course, such a problem disappears when both the donor and the recipient of the Daltonian-transfer are suitably located above the threshold. It should also be remarked here that Proposition 1 does *not* extend to general filtral preorders. As a matter of fact, if the relevant filter has two or more generators, transferring an opportunity may *destroy* heights, ruling out the possibility to obtain a profile from another one by means of a finite sequence of simple transfers. This point is easily checked by the following counterexample:

Example 3 *Let $X = \{x, y, z, u\}$ and $\mathcal{B}_F = \{\{x\}, \{y, z, u\}\}$. Let $\mathbf{A} = (A_1 = \{x, y, z\}, A_2 = \{y, z, u\}, A_3 = \emptyset, A_4 = \emptyset)$ and $\mathbf{B} = (B_1 = \{y, x\}, B_2 = \{x, y\}, B_3 = \{z\}, B_4 = \{u\})$ be two opportunity profiles, such that $\cup_i A_i = \cup_i B_i$. By Definition 3, $h_{\succcurlyeq_F}(\mathbf{A}) = (3, 1, 0, 0)$ and $h_{\succcurlyeq_F}(\mathbf{B}) = (2, 2, 0, 0)$. Therefore $\mathbf{A} \succ^M_F \mathbf{B}$. Nevertheless, transferring $\{x\}$ either directly or indirectly from A_1 to e.g. A_2, which is the only way to enhance A_2's height, would result in a new profile $\mathbf{A}^\circ$ such that $h_{\succcurlyeq_F}\left(\mathbf{A}^\circ\right) = (0, 2, 0, 0)$ hence in a net loss in the height-mass.*

---

[15] The foregoing Proposition 1 also holds for arbitrary (i.e. non-principal) filtral preorders, provided that the domain is restricted to partitional profiles, namely to opportunity profiles whose components are partitions of X. The same proof, we provide for Proposition 1, applies almost verbatim with a few slight adjustments.

Let us now proceed in our search for a SIFP-counterpart of the Theorem 1. In order to pursue this aim, we have to focus on the class of real-valued functions which preserve SIFP-induced majorization preorders.

Definition 10 (Real-valued $\succcurlyeq_F^M$-monotonic functions) *Let F be an order filter of $(\wp(X), \supseteq)$ and $\succcurlyeq_F^M$ the majorization preorder on $(\wp(X))^N$ induced by SIFP $\succcurlyeq_F$ as defined above. Then a real-valued function*

$$f : (\wp(X))^N \longrightarrow \mathbb{R}$$

is $\succcurlyeq_F^M$-monotonic *on domain $D \subseteq (\wp(X))^N$ if and only if for any $\mathbf{A}, \mathbf{B} \in D$*

$$f(\mathbf{A}) \geq f(\mathbf{B}) \qquad \text{whenever } \mathbf{A} \succ_F^M \mathbf{B}.$$

Real-valued $\succcurlyeq_F^M$-monotonic functions are simply the SIFP-counterparts of so-called Schur-convex functions (see e.g. Marshall and Olkin (1979) chapter 3 and footnote 16). Since the latter term is widely regarded as misleading, here we found it appropriate to opt for our more transparent if clumsier label.

We now establish a close SIFP-analogue to a well-known theorem from Hardy, Littlewood and Polya. Let us start with a useful characterization of real-valued $\succcurlyeq_F^M$-monotonic functions which also mimics a well known result on Schur-convex functions (see e.g. Lemma 3.A.2 in Marshall and Olkin (1979)).

Lemma 1 *Let F be a principal order filter*[16] *of $(\wp(X), \supseteq)$,*

$$h_{\succcurlyeq_F} : (\wp(X))^N \to \mathbb{Z}_+^N$$

the $\succcurlyeq_F$-induced height function as defined above, and $\varphi : \mathbb{Z}_+^N \to \mathbb{R}$. Then

$$f = \varphi \circ h_{\succcurlyeq_F}$$

is $\succcurlyeq_F^M$-monotonic on $[(H_{\succcurlyeq_F}^+)^{-1}]\downarrow$[17] *if and only if for all $\mathbf{z} \in \mathbb{Z}_+^N \cap H_{\succcurlyeq_F}^+$ such that $\mathbf{z} = \mathbf{z}\downarrow$, and $k = 1, ..., n-1$ the value*

$$\varphi(z_1, ..., z_{k-1}, z_k - \Delta, z_{k+1} + \Delta, z_{k+2}, ..., z_n)$$

is non-increasing in $\Delta \in \mathbb{Z}$ provided that:

1. $0 \leq \Delta \leq \min\{z_{k-1} - z_k, z_{k+1} - z_{k+2}\}, \quad k = 1, ..., n-2;$

[16] In fact, it is easily checked that Lemma 1 holds for *any* order filter of $(\wp(X), \supseteq)$.

[17] $[(H_{\succcurlyeq_F}^+)^{-1}]\downarrow$ is the $h_{\succcurlyeq_F}$-counter-image of the positive height span of $(\wp(X), \succcurlyeq_F)$.

2. $0 \leq \Delta \leq (z_{n-2} - z_{n-1})$.

Then, a version of the characterization of the majorization preorder in terms of Schur-convex functions is provided by the following:

Proposition 3 *Let F be a principal order filter*[18] *of $(\wp(X), \supseteq)$, $h_{\succcurlyeq_F}$ the $\succcurlyeq_F$-induced height function as defined above, and $\mathbf{A}, \mathbf{B} \in (\wp(X))^N$ with $\cup_{i=1}^n A_i = \cup_{i=1}^n B_i = X$ such that*

$$(\varphi \circ h_{\succcurlyeq_F})(\mathbf{A}) \geq (\varphi \circ h_{\succcurlyeq_F})(\mathbf{B})$$

for any $\varphi : \mathbb{Z}_+^N \rightarrow \mathbb{R}$ such that $\varphi \circ h_{\succcurlyeq_F}$ is a $\succcurlyeq_F^M$-monotonic function on $[(H_{\succcurlyeq_F}^+)^{-1}] \downarrow$. Then,

$$\mathbf{A} \succ_F^M \mathbf{B}.$$

Altogether, Propositions 1 and 2 entail the following:

Theorem 5 (Savaglio and Vanucci (Jet, 2007)) *Let F be a principal order filter of $(\wp(X), \supseteq)$, and $\mathbf{A}, \mathbf{B} \in (\wp(X))^N$ two opportunity profiles such that $\{h_{\succcurlyeq_F}(\mathbf{A}), h_{\succcurlyeq_F}(\mathbf{B})\} \subseteq H_{\succcurlyeq_F}^+$. Then, the following statements are equivalent:*

1. $\mathbf{A} \succ_F^M \mathbf{B}$;

2. There exist a $\succcurlyeq_F$-*Daltonian transfer operator* $\Im$ and a positive integer k such that $\mathbf{B} \in \Im^{(k)}(\mathbf{A})$

3.

$$(\varphi \circ h_{\succcurlyeq_F})(\mathbf{A}) \geq (\varphi \circ h_{\succcurlyeq_F})(\mathbf{B})$$

for any $\varphi : \mathbb{Z}_+^N \rightarrow \mathbb{R}$ such that $\varphi \circ h_{\succcurlyeq_F}$ is a $\succcurlyeq_F^M$-monotonic function on $[(H_{\succcurlyeq_F}^+)^{-1}] \downarrow$.

Thus, Theorem 5 is an opportunity-profile counterpart to the HLP theorem on inequality measurement as required.[19]

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[18] In fact, it is easily checked that the present Proposition holds for *any* order filter of $(\wp(X), \supseteq)$.

[19] It should be noticed that by taking $F = \wp(X)$ Theorem 2 specializes to a version of the similar characterization result of Ok and Kranich (1998).

0.4 A bridge between two research trends

Reliance on the set-inclusion order as implied by SIFPs, however, is only satisfactory when at least some of the relevant resources are **public**, or at least **non-rival** goods. Savaglio and Vannucci (Jet, 2007) suggest one way to confirm the foregoing result while avoiding such a disturbing restriction.

- If individual endowments are modelled via multisets[20] rather than set, then the items in the basic set X could be rivalrous and excludable objects, namely as **pure private goods**.

- Thus, the very same problem considered above can be addressed starting from the strict dominance order for *multisets*, as augmented with a threshold, a sort of multidimensional (opportunity) poverty line below which each opportunity set is indifferent to the null set. Hence, in order to extend the basic results of the literature on inequality measurement in the multidimensional context of opportunity profiles, we provide a suitable reformulation of the Pigou-Dalton transfer principle and of the notion of Schur convexity in a multipartition context. A counterpart of the classic HLP characterization of the Lorenz preorder on finite multipartitions via simple Pigou-Dalton transfers is provided in the multiset framework.

- It is worth noticing that a partition of multisets, or *multipartition*, is a mathematical notion that mimics a multivariate distribution and that can be represented as a rectangular matrix with integer entries whose generic row i denotes the assignment of the annual vector of goods to the i-th agent (see 5 below). Such a matrix, induced by finite multiset-partitions, is exactly the framework used to study multidimensional inequality, i.e. the disparity of a population of N individuals distinguished for several attributes.

[20] A finite *multiset* on X is a function $m : X \to \mathbb{Z}_+$ such that $\sum_{x \in X} m(x) < \infty$. A partition of multiset m -or *multipartion* of m- on population N is a profile $\mathbf{m} = \{m_i\}_{i \in N}$ of multisets on X, such that for any $x \in X$: $\sum_{i \in N} m_i(x) = m(x)$.

$$
\begin{array}{cc}
 & \quad x \quad y \; ... \; w \; ... \quad z \qquad \longleftarrow \text{goods} \\
\text{people} & \overbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad}^{} \\
\downarrow & \\
\mathbf{m} = & \begin{bmatrix}
m_1(x) & m_1(y) & ... & m_1(z) \\
. & . & & . \\
. & . & m_i(w) & . \\
. & . & & . \\
. & . & . & . \\
m_n(x) & ... & . & m_n(z)
\end{bmatrix}
\end{array}
\tag{5}
$$

Thus, the foregoing approach provides a majorization preorder of multi-profiles of goods that extends the classic unidimensional analysis of income inequality to a finite multivariate context.

0.4.1 Setting

- "Given two distribution matrices $\mathbf{m}$ and $\mathbf{m}^{'}$, which one contains the lower level of disparity?".

To answer the question, we generalize some suitable unidimensional dominance criteria to the multidimensional case. In particular, we generalize, as mentioned above, the notion of Lorenz preorder to that of Lorenz preorder of partitions of finite multisets, defined with the reference to a preorder of sets of goods as induced by strict dominance and augmented with a threshold. Then, in order to proceed with our analysis, let M_X be the set of all multisets on X and define the natural componentwise (strict) order $>$ on M_X as follows: for any m, $m^{'} \in M_X$, $m > m^{'}$ if and only if $m(x) > m^{'}(x)$ for any $x \in X$. In particular, for any $m^* \in M_X$, we may consider the subposet $\mathcal{M}_{m^*} = (M_{X,m^*}, >)$ of the poset $\mathcal{M} = (M_X, >)$, where $M_{X,m^*} = \{m \in M_X : m > m^* \text{ or } m = m^*\}$.

Definition 11 (Majorization) *Let* $\mathbf{m}, \mathbf{m}' \in \Pi_m^N$ *be two profiles of individual endowments of goods,* F *an order filter of* $(M_{X,m^*}, >)$*,* $\succcurlyeq_F$ *the corresponding filtral preorder on* M_{X,m^*}*, and* $h_{\succcurlyeq_F}$ *the* $\succcurlyeq_F$*-induced height function* *on* M_{X,m^*}. *Then, we say that* $\mathbf{m}$ *majorizes* $\mathbf{m}'$, *denoted* $\mathbf{m} \succcurlyeq_F^{maj} \mathbf{m}'$, *if:*

$$h_{\succcurlyeq_F}(\mathbf{m}) = (h_{\succcurlyeq_F}(m_1), ..., h_{\succcurlyeq_F}(m_n)) \succcurlyeq^{maj} (h_{\succcurlyeq_F}(m'_1), ..., h_{\succcurlyeq_F}(m'_n)) = h_{\succcurlyeq_F}\left(\mathbf{m}'\right),$$

namely:

$$\begin{aligned} \sum_{i=1}^{k} h_{\succcurlyeq_F}(m_i) &\geqslant \sum_{i=1}^{k} h_{\succcurlyeq_F}(m'_i) \qquad k = 1, ..., n-1, \\ \text{and } \sum_{i=1}^{n} h_{\succcurlyeq_F}(m_i) &= \sum_{i=1}^{n} h_{\succcurlyeq_F}(m'_i), \end{aligned}$$

whenever the height vectors are arranged in non-increasing order.

Now, in order to extend the basic results of the literature on unidimensional inequality measurement to our multidimensional context of profiles of individual endowments of private goods, we must provide a suitable reformulation of the Pigou-Dalton transfer principle. Thus, let us state the notion of *transfer* with respect to height-extensions of DFPs, by first defining a *transfer operator* as follows:

Definition 12 *A transfer operator on* Π_m^N *is a nonempty correspondence* $\Im : \Pi_m^N \rightrightarrows \Pi_m^N$ *such that*

$$\forall (\mathbf{m}, \mathbf{m}') \in \Pi_m^N \times \Pi_m^N, \ \mathbf{m}' \in \Im(\mathbf{m}).$$

Then, a transfer operator is a transformation which leaves the set of all total alternatives/goods in $\mathbf{m}$ and $\mathbf{m}'$ unchanged. Next, we define a notion of *minimal* transfer:

Definition 13 *Let* $\mathbf{m}, \mathbf{m}' \in \Pi_m^N$ *be two profiles of individual endowments of goods,* F *a principal order filter of* $(M_{X,m^*}, >)$ *with basis* $\mathcal{B}_F = \{b\}$ *and* $i, j \in N$ *such that* $m_i \succ_F m_j$, $h_{\succcurlyeq_F}(m_i) > h_{\succcurlyeq_F}(m_j) + 1$ *such that:*

$$\begin{aligned} m'_i(x) &= m_i(x) - 1 \text{ for any } x \in X \\ m'_j(x) &= m_j(x) + 1 \text{ for any } x \in X \\ \text{and } m'_l(x^*) &= m_l(x^*) \text{ for any } l \neq i, j, \text{ and } x^* \in X \end{aligned} \tag{6}$$

Then $\mathbf{m}'$ *is said to arise from* $\mathbf{m}$ *through a transfer (from richer* i *to poorer* j*).*

By analogy with the Pigou-Dalton principle, we also require that transfers of goods be not large enough to reverse the relative height-induced positions of the donor and recipient, namely:

Definition 14 *A transfer operator* $\Im$ *is said to be:*

(i) weakly rank-monotonic *w.r.t* $\succcurlyeq_F$ *if and only if it does not cause height-reversals i.e. for any* $\mathbf{m}, \mathbf{m}' \in \Pi_m^N$ *and any* $i, j \in N$*, if*

$$\mathbf{m}' \in \Im(\mathbf{m}),\ m_i' \neq m_i,\ m_j' \neq m_j$$

and

$$h_{\succcurlyeq_F}(m_i) \geqslant h_{\succcurlyeq_F}(m_j)$$

then

$$h_{\succcurlyeq_F}(m_i') \geqslant h_{\succcurlyeq_F}(m_j').$$

(ii) weakly progressive *w.r.t.* $\succcurlyeq_F$ *if and only if for any* $\mathbf{m}, \mathbf{m}' \in \Pi_m^N$*:*

$$\mathbf{m}' \in \Im(\mathbf{m}),\ m_i' > m_i \text{ and } m_j > m_j'$$

entails that

$$h_{\succcurlyeq_F}(m_i) \geqslant h_{\succcurlyeq_F}(m_j).$$

(iii) weakly-equalizing *w.r.t.* $\succcurlyeq_F$ *if it is both weakly rank-monotonic w.r.t.* $\succcurlyeq_F$ *and weakly progressive w.r.t.* $\succcurlyeq_F$*.*

Moreover, in order to pursue our search for a DFP-counterpart of the HLP's celebrated result, we have to focus on the class of real-valued functions which preserve DFP-induced majorization preorders.

Definition 15 (real-valued $\succcurlyeq_F^{maj}$-isotonic functions) *Let* F *be an order filter of* $(M_{X,m^*}, >)$ *and* $\succcurlyeq_F^{maj}$ *the majorization preorder on* Π_m^N *induced by the DFP* $\succcurlyeq_F$ *as defined above. Then a real-valued function*

$$f : \Pi_m^N \longrightarrow \mathbb{R}$$

is isotonic *(wrt* $\succcurlyeq_F^{maj}$*) on domain* $D \subseteq \Pi_m^N$ *if and only if for any* $\mathbf{m}, \mathbf{m}' \in D$

$$f(\mathbf{m}) \geqslant f(\mathbf{m}') \qquad \text{whenever } \mathbf{m} \succ_F^{maj} \mathbf{m}'.$$

Finally, the use of the foregoing Definitions is clarified in the following:

Example 4 *Let us suppose that the set of available goods X is composed of six copies of good x and ten copies of good y, (i.e. $m(x) = 6$ and $m(y) = 10$), distributed over a population of three agents $\{i, j, l\}$ in order to get a partition of multiset m, namely the multiprofile:*

$$\mathbf{m} = \begin{matrix} \\ i \\ j \\ l \end{matrix} \begin{matrix} x \;\; y \\ \begin{pmatrix} 5 & 6 \\ 1 & 2 \\ 0 & 2 \end{pmatrix} \end{matrix}$$

If we consider as the basis *of the filter $\mathcal{B}_F = \{b_1, b_2\}$, where $b_1 = (b_1(x)) = 1$ and $b_2 = (b_2(y)) = 1$, then the corresponding filter-induced height function will be tantamount to $h_{\succcurlyeq_F}(\mathbf{m}) = (5, 1, 0)$. Thus, suppose that a transfer takes place from richer i to poorer l in order to get the new multidimensional distribution:*

$$\mathbf{m}' = \begin{matrix} \\ i \\ j \\ l \end{matrix} \begin{matrix} x \;\; y \\ \begin{pmatrix} 4 & 5 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \end{matrix}$$

and the corresponding $h_{\succcurlyeq_F}(\mathbf{m}') = (4, 1, 1)$. Hence, it is obvious that $\mathbf{m} \succcurlyeq_F^{maj} \mathbf{m}'$ and that $f(\mathbf{m}) \geqslant f(\mathbf{m}')$ where f is, for example, a function that simply sums the value of the heights of the multipartitions. On the contrary, if $\mathcal{B}_F = \{b_1, b_2\} = (1, 3)$, and the same transfer takes place in $\mathbf{m}$, we now have that $h_{\succcurlyeq_F}(\mathbf{m}) = (4, 0, 0)$ and $h_{\succcurlyeq_F}(\mathbf{m}') = (3, 0, 0)$, with corresponding net loss of height mass. It is worth noticing here how a careful check that numbers always vary according to transfers of goods is often required.

0.5 A HLP Theorem for finite multipartitions

Theorem 6 *Let F be a principal order filter of $(M_{X,m^*}, >)$, and $\mathbf{m}, \mathbf{m}' \in \Pi_m^N$ two opportunity profiles such that $h_{\succcurlyeq_F}(\mathbf{m}), h_{\succcurlyeq_F}(\mathbf{m}') \in H^+_{\succcurlyeq_F}$. Then, the following statements are equivalent:*

1. $\mathbf{m} \succ_F^{maj} \mathbf{m}'$;

2. There exist a $\succcurlyeq_F$-*weakly equalizing transfer operator* $\Im$ and a positive integer k such that $\mathbf{m}' \in \Im^{(k)}(\mathbf{m})$

3.
$$(\varphi \circ h_{\succcurlyeq_F})(\mathbf{m}) \geqslant (\varphi \circ h_{\succcurlyeq_F})(m')$$
for any $\varphi : \mathbb{Z}_+^N \to \mathbb{R}$ such that $\varphi \circ h_{\succcurlyeq_F}$is a $\succcurlyeq_F^{maj}$-isotonic function on $[(H^+_{\succcurlyeq_F})^{-1}]$e multipartitions

To conclude, the componentwise strict dominance preorder of vectors, representing the assignment of the goods to the agents, supports a multipartition counterpart to the celebrated HLP Theorem. In a sense, Savaglio and Vannucci (Jet, 2007) provide a somewhat optimistic answer to the question: "*A lost paradise?*", raised by Trannoy (2006) and concerning the *possibility* of finding again "*the miracle of the HLP theorem*" in the multidimensional context.

Part IV

Conclusions

- The relevance of the foregoing results relies on the fact that the DFP-approach is conducive to a majorization preorder of multiprofiles of goods that extends the classic unidimensional analysis of income inequality to a multivariate context.

- Since the comparison of multidimensional distributions typically admits only a non-total preorder of individual endowments, we have suggested the possibility to rely on height-based total extensions in order to reproduce some relevant parts of the theory of majorization (or, dually, Lorenz) preorders. Indeed, we have shown that the componentwise strict preorders of vectors, representing the assignment of the goods to the agents, support a multipartition counterpart to the celebreted HLP Theorem.

- In a broad sense, we answer to the question: "*A lost paradise?*", posed by Trannoy (2006) and concerning the *impossibility* of finding again "*the miracle of the HLP theorem*" in the multidimensional context. Of course, this does not come totally for free.

  1. We first needed to use a two-steps procedure in order to compare rectangular matrices, representing the disparity of a population of N individuals distinguished for several attributes, namely multivariate ditributions of goods. Then, we adopted a very restricted version of the Pigou-Dalton principle of transfers to define a distributive profile as less even than another one.

Although our work represents a new fruitful approach to the analysis of multidimensional inequality, much more remains to be discovered, at least on the problem to compare our solution to the issue of building up a Lorenz preorder of multivariate distributions with the main results on matrix majorization existing in economic literature, but this task is best left as a possible topic for further research.

0.6 A caveat

The present work is a "cut and paste" almost verbatim of the following papers:

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