Risk, time and uncertainty: A short introduction

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Introduction

Why risk and time?

- Risk and time are essential component of social outcomes
- In particular, inequalities across time and events
- But isnt't this just multidimensional inequalities?
- No.

The perspective issue

A simple sharing problem

- Two divisible goods, 1 ($\omega_1 = 1$) and 2 ($\omega_2 = 1$)
- Two identical individuals, a and b
- $u_i(x_1, x_2) = x_1 + x_2$

P_1	a	b
1	0	1
2	1	0

P_2	a	b
1	1	1
2	0	0

In what sense could we say that P_2 is less unequal than P_1?

The perspective issue

A simple sharing problem

- One good, two equiprobable states, 1 ($\omega_1 = 1$) and 2 ($\omega_2 = 1$)
- Two identical individuals, a and b
- $u_i(x_1, x_2) = \frac{1}{2}x_1 + \frac{1}{2}x_2$

P_1	a	b
1	0	1
2	1	0

P_2	a	b
1	1	1
2	0	0

Any good reason to think that P_2 is less unequal than P_1?

The ex post perspective

Social preferences

A simple sharing problem

- Two divisible goods, 1 ($\omega_1 = 1$) and 2 ($\omega_2 = 1$)
- Two identical individuals, a and b
- $u_i(x_1, x_2) = x_1 + x_2$

P_1	a	b
1	1	1
2	0	0

P_2	a	b
1	.5	.5
2	.5	.5

In what sense could we say that P_2 is better than P_1?

Social preferences

A simple sharing problem

- One good, two equiprobable states, 1 ($\omega_1 = 1$) and 2 ($\omega_2 = 1$)
- Two identical individuals, a and b
- $u_i(x_1, x_2) = \frac{1}{2}x_1 + \frac{1}{2}x_2$

P_1	a	b
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Any good reason to think that P_2 is better than P_1?

Social risk aversion

A specific problem

Two specific issues

- Admittedly, the *ex post* (instantaneous) perspective exists.
- Social planner might care about risk, time consistency...

Consequence

The problem cannot simply be reduced to measuring multidimensional inequalities

An important problem

"The moral of this story is that simply specifying a social welfare function may not be enough to fully determine a procedure for collective decision making. One must also specify when the individuals' preferences or utility levels should be evaluated; before or after the resolution of uncertainties. The timing of social welfare analysis may make a difference. The timing-effect is often an issue in moral debate, as when people argue about whether a social system should be judged with respect to its actual income distribution or with respect to its distribution of economic opportunities"

Myerson, Econometrica, 1981 (p. 884).

Harsanyi's Theorem (1955)

Assumptions

- Individuals and society are EU: $U_i(p) = \sum_{x \in X} p(x) u_i(x)$
- Pareto
- diversity

Result

$$u_0(x) = \sum_i \lambda_i u_i(x) + \mu$$

Diamond's critique (1967)

p	a	b
1	1	0
2	1	0

q	a	b
1	1	0
2	0	1

Ex ante approach

- $V(p) = \frac{1}{2}V_a(p) + \frac{1}{2}V_b(p) = \mathbf{\frac{1}{2}}$
- $V(q) = \frac{1}{2}V_a(q) + \frac{1}{2}V_b(q) = \mathbf{\frac{1}{2}}$
- $\Rightarrow p \sim q$

Problem

Neglects inequalities in *ex ante* (expected) utilities

Setup

- 2 individuals
- 2 equiprobable states
- $x_i(s) \geq 0$: *outcome* of i in state s
- $x_i(s)$ fully measurable and interpersonnaly comparable

Remarks

- $x_i(s)$ can be e.g. income...
- but also utility of i in s (assuming it exists)...
- we might also assume Expected Utility

General idea

References

- Diamond, *JPE* (1967)
- Epstein and Segal, *JPE* (1992)

How it works

	a	b
1	$x_a(1)$	$x_b(1)$
2	$x_a(2)$	$x_b(2)$

General idea

References

- Diamond, *JPE* (1967)
- Epstein and Segal, *JPE* (1992)

How it works

	a	b
1	$x_a(1)$	$x_b(1)$
2	$x_a(2)$	$x_b(2)$

$$\longrightarrow [V(x_a), V(x_b)]$$

General idea

References

- Diamond, *JPE* (1967)
- Epstein and Segal, *JPE* (1992)

How it works

	a	b
1	$x_a(1)$	$x_b(1)$
2	$x_a(2)$	$x_b(2)$

$$\longrightarrow [V(x_a), V(x_b)] \longrightarrow W\left(V(x_a), V(x_b)\right)$$

Objection

x	a	b
1	1	0
2	0	1

y	a	b
1	1	1
2	0	0

- $V(x_a) = V(y_a)$
- $V(x_b) = V(y_b)$
- $\Rightarrow x \sim y$

Neglects *ex post* inequalities

General idea

	a	b
1	$x_a(1)$	$x_b(1)$
2	$x_a(2)$	$x_b(2)$

General idea

	a	b
1	$x_a(1)$	$x_b(1)$
2	$x_a(2)$	$x_b(2)$

$$\longrightarrow [W(x(1)), W(x(2))]$$

General idea

	a	b
1	$x_a(1)$	$x_b(1)$
2	$x_a(2)$	$x_b(2)$

$$\longrightarrow [W(x(1)), W(x(2))]$$

$$\longrightarrow V\left(W(x(1)), W(x(2))\right)$$

General idea

	a	b
1	$x_a(1)$	$x_b(1)$
2	$x_a(2)$	$x_b(2)$

$$\longrightarrow [W(x(1)), W(x(2))]$$

$$\longrightarrow V\left(W(x(1)), W(x(2))\right)$$

Ex post inequality aversion

W quasi-concave

A naive solution

Ex post welfare

$W(x(s)) = \frac{1}{n}\sum_i \varphi(x_i(s))$, φ concave.

Expected welfare

$V(x) = \sum_s \pi(s)\left(\frac{1}{n}\sum_i \varphi(x_i(s))\right)$

Problem (Fleurbaey, 2009)

- No inequalities: $x_i(s) = x_j(s) = x(s)$ for all i, j, s
- Expected utility: $\sum_s \pi(s)x(s)$
- Expected welfare: $\sum_s \varphi(x(s))$
- How to justify this extra risk aversion (given equality)?

Ex post ordering

Continuous complete ordering on social prospects: $\succcurlyeq$

$$(x_a(s), x_b(s)) \succcurlyeq_p (y_a(s), y_b(s)))$$

$$\Leftrightarrow \quad \begin{array}{c|cc} & a & b \\ \hline 1 & x_a(s) & x_b(s) \\ 2 & x_a(s) & x_b(s) \end{array} \succcurlyeq \begin{array}{c|cc} & a & b \\ \hline 1 & y_a(s) & y_b(s) \\ 2 & y_a(s) & y_b(s) \end{array}$$

Expected equally-distributed equivalent

Fleurbaey, "Assessing risky social situations", 2009.

Equally distributed equivalents

$$(e(x(s)), e(x(s))) \sim_p (x_a(s), x_b(s))$$

Expected equally distributed equivalent

- $V(x) = \sum_s \pi(s) e(x(s))$
- Example: $V(x) = \sum_s \pi_s \varphi^{-1}\left(\frac{1}{n}\sum_i \varphi(x_i(s))\right)$

Properties

- behaves like *ex ante* criteria when risk does not generate inequalities
- behaves like *ex post* criteria otherwise

Axioms

Axiom (Dominance)

$x(s) \succcurlyeq_p y(s)$ *for all* $s \Rightarrow x \succcurlyeq y$

Avoiding the drawback of the naive solution:

Axiom (Weak Pareto for Equal Risk)

If $x_i = x_j$ *for all* i, j *then:*

$$x \succ y \Leftrightarrow \sum_s x_i(s) > \sum_s y_i(s)$$

Axiom (Weak Pareto for No Risk)

If $x_i(s) = x_i(s')$ *for all* i, s, s' *then:*

$$[x_i(s) > y_i(s) \text{ for all } i] \Rightarrow x \succ y$$

Combining ex ante and ex post approaches

x	a	b
1	0	0
2	1	1

y	a	b
1	0	1
2	1	0

z	a	b
1	1	0
2	1	0

"Natural" ordering

$$x \succ y \succ z$$

Ben Porath, Gilboa, Schmeidler (*JET*, 1997)

- $V * W = V(W(x(1), W(x(2)) : x \succ y \sim z$
- $W * V = W(V(x_a), V(x_b)) : x \sim y \succ z$

Combining ex ante and ex post approaches

x	a	b
1	0	0
2	1	1

y	a	b
1	0	1
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"Natural" ordering

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- $V * W = V(W(x(1), W(x(2)) : x \succ y \sim z$
- $W * V = W(V(x_a), V(x_b)) : x \sim y \succ z$
- $\alpha V * W + (1 - \alpha) W * V$: $x \succ y \succ z$

Axiomatic foundation: Gajdos and Maurin, *JET*, 2004.

Ex ante ordering

Continuous complete ordering on social prospects: $\succcurlyeq$

$$(x_a(1), x_a(2)) \succcurlyeq_a (y_a(1), y_a(2)))$$

$$\Leftrightarrow \quad \begin{array}{c|cc} & a & b \\ \hline 1 & x_a(1) & x_a(1) \\ 2 & x_a(2) & x_a(2) \end{array} \succcurlyeq \begin{array}{c|cc} & a & b \\ \hline 1 & y_a(1) & y_a(1) \\ 2 & y_a(2) & y_a(2) \end{array}$$

Axioms

Axiom (Monotonicity)

$$[x_i(s) \geq y_i(s) \text{ for all } i, s] \Rightarrow x \succsim y$$

Axiom (Dominance)

$$\left.\begin{array}{l} x_i \succsim_a y_i, \ \forall i \\ x(s) \succsim_p y(s), \ \forall s \end{array}\right\} \Rightarrow x \succsim y$$

Axiom (Homogeneity)

$x \succsim y \Rightarrow \lambda x \succsim \lambda y, \ \forall \lambda > 0$

Axioms

Axiom (Conditional Dominance)

$$\begin{array}{c|cc} & a & b \\ \hline 1 & \mu(1)\lambda_a & \mu(1)\lambda_b \\ 2 & \mu(2)\lambda_a & \mu(2)\lambda_b \end{array} \quad \succcurlyeq \quad \begin{array}{c|cc} & a & b \\ \hline 1 & \nu(1)\lambda_a & \nu(1)\lambda_b \\ 2 & \nu(2)\lambda_a & \nu(2)\lambda_b \end{array}$$

$$\Leftrightarrow \quad \begin{array}{c|cc} & a & b \\ \hline 1 & \mu(1) & \mu(1) \\ 2 & \mu(2) & \mu(2) \end{array} \quad \succcurlyeq \quad \begin{array}{c|cc} & a & b \\ \hline 1 & \nu(1) & \nu(1) \\ 2 & \nu(2) & \nu(2) \end{array}$$

Result

- $I(x) = \Psi(W * V, V * W)$
- $W * V(x) = V * W(x) \Rightarrow I(x) = W * V(x)$
- $W * V(x) < V * W(x) \Rightarrow W * V(x) < I(x) < V * W(x)$
- Ψ, W and V homogeneous
- V and W $\propto$ unique
- given (V, W), Ψ unique

Beyond consequentialism

Work in progress (with Marc Fleurbaey): relaxing consequentialism

The idea

Diamond's critique revisited

- The allocation *process* matters
- In state s, what would have happened in state s' might be relevant
- Individual i utility in state s might depend on $(x_i(1), \cdots, x_i(s))$
- Relaxing consequentialism?

Idea

- replace $\succcurlyeq_p$ by $\succcurlyeq_s$
- $\succcurlyeq_s$ compares x and y if s
- $\succcurlyeq_s$ is not constrained to depend only on $x(s)$
- $\succcurlyeq$ satisfies Pareto wrt $\{\succcurlyeq_s\}_s$

The "Result"

Structural assumptions

$x \succcurlyeq y \Leftrightarrow \sum_s \Psi\left(\varphi_s(x)\right) \geq \sum_s \Psi\left(\varphi_s(y)\right)$

Result

$(\succcurlyeq, \{\succcurlyeq_s\}_s)$ satisfies State Neutrality, Independence of the Utility of the Sure, Ex Post Individualism and Anonymity, then

(i) $\succcurlyeq_s$ can be represented by: $\sum_i \varphi_s(x_i)$;

(ii) $\succcurlyeq$ can be represented by: $\sum_s \psi\left(\sum_i \varphi_s(x_i)\right)$.

Moreover, either:

(i) $\psi(x) = \alpha x + \beta$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$, or

(ii) $\psi(x) = e^{\alpha x}$ for some $\alpha > 0$.

Setup

Bommier and Zuber: "The Pareto principle of optimal inequality" (2009)

Alternatives

- Z: pure outcomes
- S: state space
- $\mathscr{F} = \{f : S \to Z\}$: act
- Two periods:
  1. $Y = Z \times \mathscr{F}$: choice set in period 1
  2. Z: choice set in period 2

preferences

Individuals

1. $U : Y \to \mathbb{R}$
2. $U_{z,} : Z \to \mathbb{R}$

$\mathscr{U} = \{(U, U_z)|z \in Z\}$: process of preferences

Observer

- Social evaluation functions
  1. $W : Y \to \mathbb{R}$
  2. $W_z : Z \to \mathbb{R}$
- Interpersonal utility functions:
  1. $U : Z \times \{1, 2\} \to \mathbb{R}$
  2. $U_z : Y \times \{1, 2\} \to \mathbb{R}$

Social observer: $(\mathscr{W}, \mathscr{U})$

Comparative inequality aversion

Definition (More unequal than)

Prospect x is more unequal than prospect y ($x \rhd y$) if

$$U(x,i) \leq U(y,i), U(y,j) \leq U(x,j)$$

with a strict inequality

Definition (More inequality averse than)

Social Observer A is more inequality averse than SO B iff for all social prospect x:

$$[y \rhd_A x \text{ and } W^A(y) \geq W^A(x)] \Rightarrow [y \rhd_B x \text{ and } W^B(y) \geq W^B(x)]$$

Axioms

Time Consistency

$[U_z(f(s)) \geq U_z(f'(s)), \forall s] \Rightarrow U(z, f) \geq U(z, f')$

Pareto

- $U(\cdot, i)$ represents i's preferences
- W only depends on $U(\cdot, 1)$ and $U(\cdot, 2)$

Reversibility

A state of the world s is socially revertible if, whatever z is obtained in first period, $(z, f(s))$ do not fully determine the individuals welfare ranking.

Result

Proposition

Assume individuals are time consistent. Consider two paretian social observers A and B who are time consistent. If

1. *there exists a socially invertible state for A*
2. *A is at least as inequality averse than B*

then

A and B have same preferences in period 2.