Electoral Competition and Redistribution

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Abstract

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Introduction

What are the fondations of individual preferences for redistribution ? Self Interested agents driven by their own welfare, social norms, altruism, prospect of upward mobility,...

Politicians are not benevolent dictators : the act with respect to their own personal agenda . Here we follow the Downsian view that they are purely opportunistic

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Outline of the Presentation

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A General Model

A. Economy, Agents and Policies

B. Candidates and Elections

C. Game : Downsian Competition and Discontinuous Two-Player Zero-Sum Games

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Economy, Agents and Policies

We postulate a set of *economic types* T_e, where the generic *citizen* is denoted by $t_e \in T_e$. A citizen's economic type may be thought of as a vector of traits, which characterizes his economic preferences and endowments. The population of citizens is characterized by a probability measure F on T_e. Thus if S is a measurable subset of T_e, then $F(S)$ is the fraction of the population who have traits $t_e \in T_e$. F could be a probability distribution with finite or infinite support. We could alternatively describe the population of citizens by the set of names N which can be taken as the interval $[0, 1]$, a measure λ on N and a measurable mapping from N into T_e. We could of course consider $N = T_e$ but in many settings the relevant information to consider is not the name of an individual per se.

The second key primitive is a set of *policies* A. Every citizen has a preference over policies, represented by a utility function $v : A \to \Re$. We denote the utility function of a citizen of type t as $v(.,t_e)$. In most of the problems, a policy is a vector of variables under the control of the public sector like for instance taxes, public expenditures or allocation of specific budgets. Then, the function $v(,t_e)$ represents the indirect utility function of a citizen of economic type t_e : this means that it accounts for the equilibrium effects of the public policy. This may be problematic when there are multiple equilibria compatible with a policy but in the very stylized models considered here this concern will not appear.

I illustrate this general setting with four examples that will be used extensively in our presentation.

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Example 1 : Unrestricted Redistributive Politics

In the simplest version of the problem, there is an exogenous endowment M of a single consumption good to be distributed among the citizens. A policy is therefore a function $a : N \rightarrow \Re_+$ such that $\int_N a(n)d\lambda(n) = M$. Except for the feasibility constraint, this set entails no restrictions.

$$A = \left\{ a : N \rightarrow \Re_+ \text{ such that} : \int_N a(n)d\lambda(n) = M \right\}$$

Here it is convenient to consider $T_e = N$. Further, under the assumption that preferences are monotonic and selfish we may assume that $v(a,t_e) = \widetilde{v}(a(t_e),t_e)$ where $\widetilde{v}$ is increasing in its first argument.

Some authors consider as primitives individual endowments in this commodity. If we denote by $e(n)$ the endowment of n, then :

$$A = \left\{ a : N \rightarrow \Re_+ \text{ such that} : \int_N a(n)d\lambda(n) = 0 \text{ and } a(n) \geq -e(n) \text{ for all } n \in N \right\}$$

and $v(a,t_e) = \widetilde{v}(a(t_e) + e(t_e),t_e)$.

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Example 2 : Restricted Redistributive Politics

In some applications, like for instance redistribution across age groups or regions, the set of policies may be reduced as the result of *measurability* constraints. For instance, when the population is partitioned into finitely many groups $\left\{N^1, N^2, ..., N^J\right\}$according to age, geography or any other variable, a distributive policy may be unable (or not allowed) to discriminate between two citizens belonging to the same group i.e. :

$$A = \left\{ \begin{array}{c} a : N \to \Re_+ \text{ such that} : \int_N a(n) d\lambda(n) = M \\ \text{and } a(n) = a(n') \text{ for all } n, n' \in N^j \text{ for some } k = 1, .., J \end{array} \right\}$$

In that case, the set A can alternatively be described as :

$$A = \left\{ a \in \Re_+^J : \sum_{j=1}^{J} n^j a^j = M \right\}$$

where $n^j \equiv \lambda(N^j)$ for all $j = 1, ..., J$.

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Example 3 : Second(Third,...) Best Redistributive Politics

In some other applications, on top of which redistribution across income groups, the set of first best policies is restricted by *incentive* constraints. Consider for instance redistribution policies in an economy a la Mirrlees. There are two goods in the economy : a unique consumption good and labor (leisure). The consumption good is produced with labor through a constant returns to scale technology. The economic type of an agent is his productivity (talent) : a citizen with type t_e working for one unit of time produce t_e units of the consumption good. Otherwise, the citizens are assumed to have the same tastes described by a utility function $u(c, l)$ over the pairs (c, l) where c denotes the consumption level and l the number of hours spent in working. A feasible allocation is here a function $x : T_e \equiv \Re_+ \to \Re_+^2$ mapping t into a pair $(c(t_e), l(t_e))$ such that : $\int_{T_e} c(t_e) dF(t_e) = \int_{T_e} t_e l(t_e) dF(t_e)$. However if the economic type of a citizen is not observable, the set of policies is a proper subset of the set of feasible allocations. Precisely, when only the gross income $y(t_e) \equiv t_e l(t_e)$ is observable, the set of policies A is the set of functions $a : T_e \to \Re_+^2$ mapping t_e into a pair $(c(t_e), y(t_e))$ such that $\int_{T_e} c(t_e) dF(t_e) = \int_{T_e} y(t_e) dF(t_e)$ and $u((c(t_e), \frac{y(t_e)}{t_e}) \geq u((c(t'_e), \frac{y(t'_e)}{t_e})$ for all $t'_e \in T_e$. A policy can be alternatively presented as a (possibly non linear) income taxation scheme. Further, $v(a, t_e) = u(c(t_e), \frac{y(t_e)}{t_e})$.

Some authors have even considered subsets of the set of general income taxation schemes by imposing some extra mathematical structure on the functions like linearity or other more complicated forms of parametric restrictions. In this *third best* framework, the set of policies A becomes a subset of some multidimensional (possibly unidimensional) Euclidean space. This specific setting has been explored in many papers.

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Example 4 : Redistributive Politics and Public Spending

In all preceding examples, a policy was exclusively a program for redistributing resources across citizens, possibly subject to some constraints. We may add to that aspect another dimension describing *public spending*. In that respect, the simplest extension of example 1, proposed by Lizzeri and Persico (2001) would consist in adding the possiblity for the public sector to confiscate the totality of private resources in to produce a (Samuelsonian) public good i.e. the set of policies is the set of example 1 to which is added a single policy : producing the public good after taxing all private resources. Preferences for public goods are described by an extra economic type; Lizzeri and Persico assume that the citizens are all alike in the way they value public goods.

This model is of course highly stylized but more general models in the same vein can be considered. The interest of such extended setting is to introduce a trade off between redistributive concerns and efficiency considerations via public good provision.

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Candidates and Elections

We consider an election in which there are K candidates for a single indivisible office and the voters vote according to a *rank-scoring* election rule. Such an electoral system can be characterized by an ordered sequence of K numbers denoted here by $s_1, s_2, ..., s_K$, where $1 = s_1 \geq s_2 \geq .... \geq s_K = 0$. In the election each voter must indicate a ranking of the K candidates on a ballot; then this ballot gives the top-ranked candidate s_1 points, the second-ranked candidate s_2 points, and so on. Each candidate's *score* is the average of the points received by all voters. The winner of the election is the candidate with the highest score.

Plurality voting (or single nontransferable vote (SNTV)) is a rank scoring rule in which $s_1 = 1$ and $s_k = 0$ for all $k > 1$. More generally, for any V between 1 and K, a rule where each voter must distribute V noncumulative votes is represented by setting $s_1 = s_2 = ... = s_V = 1$ and $s_k = 0$ for all $k > V$. Negative-plurality voting corresponds to the case where $V = K-1$. Borda rule is a rank-scoring rule in which $s_k = \frac{K-k}{K-1}$ for all k. Given any scoring rule, $\overline{s}$ will denote the average of the ranking points that a voter can give, so

$$\overline{s} = \frac{\sum_{k=1}^{K} s_k}{K}$$

While general, this model of multicandidate competition does not cover all the situations of interest. First, there are some important electoral systems which are not rank-scoring.

For instance *approval voting* where each voter indicates approval of a set of candidates on the ballot is not a rank-scoring rule. Nor is plurality with runoff or single transferable vote (STV).

Second, the assumption that there is a unique seat summarizing political power is highly stylized; a full description of the microstructure of the political institutions and the various branches of the state and their interaction would however be very demandind. As a fist step in the direction, we could extend the framework by considering that we have multiseat elections. For instance, in a pure parliamentary system (no other election than the election of the members of a single chamber) there are L seats to be allocated if L is the number of representatives. In such context, *proportional electoral systems* have received a lot of attention. If we interpret the K competitors as parties instead of candidates, proportional systems allocate the L seats in proportion to the votes (scores) obtained by the party; given integer problems perfect representation is not possible and there are many ways to define representation in electoral systems. Note also that from the perspective of the voters, the voting decision gets complicated as he has to figure out what would be the consequences for him of any possible composition of the chamber. This means that he must have in mind a *model mapping chamber compositions into policies.* Most of the authors discussing this issue assume the following mapping : the policy implemented will be the policy announced by party k during the election campaign with probability $\frac{L_k}{L}$ where L_k denotes the number of seats won by party k.

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Another electoral system consists in partioning the population into L districts : each district elects one representative according to one of the rules discussed before. We can take the same mapping as before. Another mapping which is often considered consists in assuming that the policy of the party which won the largest number of seats is implemented.

These reduced forms are very crude descriptions of the political sphere and, during the talk, we will discuss the insights that could come however from these approximations.

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Game : Downsian Competition

In this manuscript, we will focus our attention on the following game. The actors of this game will be the citizens (voters) and the K candidates (parties). The game is sequential and has three stages

Stage 1 : **Preelectoral Politics (Electoral Campaign)** : Each candidate $k \in \{1, 2, ..., K\}$ announces a policy $\widetilde{a}_k$

Stage 2 : **Elections** : Voters vote according to one of the electoral systems discussed previously for one or possibly many seats

Stage 3 : **Postelectoral politics** : Here we will considered a reduced form mapping the political platforms anounced during the campaign and the outcomes of elections (stage 2) into policies.

As uncertainty will be present, nature will be introduced as an additional player. We will explain soon how the game has to be accurately defined in that respect. For the moment, let us turn our attention to the definition of the payoff of the various actors. Concerning candidates (parties) we will assume that there are Downsian in that they have no preferences over the policy that will be ultimately implemented; policies are strategic instruments to " win" the election. As win can be understood in many ways, the exact definition of the payoff may be subject to different interpretations. But certainly, it is reasonable to assume that the payoff is weakly monotonic with respect to the number of seats won during the election. If the number of seats is itself weakly monotonic with respect to the score of the party during the election, then the the score of the party is a good proxy for its payoff.

The payoff of any voter depends on two arguments : the policy a that is ultimately implemented and the outcome of the election represented here by the symbol E. The impact of the first argument depends on the economic type t_e of the voter through the expression $v(a, t_e)$. The impact of the second argument depends on the political type of the voter; this political type denoted by t_p belongs to the set T_p of possible political types. The payoff of a voter with type $t = (t_e, t_p)$ will be denoted $u(v(t_e, a), E, t_p)$. The dependence of the payoff of the voter with respect to the result of the election per se is traditionally justified by the fact that there are many ideological dimensions in the policy of a party that is not represented in policy a. More precisely the all policy could be defined as having two parts : a pliable part which would be a and another part that would be fixed. This second part may also represent some important aspect valence dimensions of the party like for instance competence or immunity to corruption. The political type may contain *private value* components but also *common value* components in particular if valence dimensions are accounted for. The population of citizens with economic type t_e is characterized by a conditional probability measure $G(t_e)$ on T_p.

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The game is solved backward. As we mention, the last stage is assumed to be already solved. In the second stage the active players are the voters. In some cases, the strategic analysis of that stage may be quite difficult as sincere voting (when this expression makes sense) may be different from optimal voting. The calculation of the Nash equilibria is a tricky issue. However in the case where $K = 2$, sincere voting is a dominant strategy. When the second stage is solved, we examine the first stage where the active players are the candidates 'parties) : parties compete against each other through campaigns anticipating the continuation of decisions made at that stage. If we denote by $E(\widetilde{a})$ the election outcome following the profile of announcements $\widetilde{a} \equiv (\widetilde{a}_1, \widetilde{a}_2, ..., \widetilde{a}_K)$, we have to analysed the reduced game where the players are the K candidates :each candidate k selects his (pure) strategy in A and derives a payoff equal to $\pi_k(E(\widetilde{a}))$ when the profile of pure strategies is $\widetilde{a}$.

A last qualification is needed as players (candidates and voters) may have make their choices with limited information about some relevant aspects of this strategic environment. Here, we will limit our attention to the uncertainty faced by the candidates in the first stage. They may not know with certainty the population of citizens (voters); for instance they may not know the economic or (and) political type of the citizens. In this manuscript, we will assume that they know the distributions F and $G(| \, t_e)$ of economic and political types. This does not mean however that the uncertainty is resolved when there is a continuum of types as the type may have a common value component and the law of large numbers may not apply.

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K=2

Except for the presentation of Myerson 's seminal contribution (1993), the manuscript will focus on the case where two candidates (parties) compete for office(s). If there is a single seat, all electoral systems coincide. If there are many seats, there are of course different electoral systems. We will focus on two of them. "Perfect" representation corresponds to the case where there is a single electoral district, voters vote for one of the two parties, and the percentage of votes obtained by a party determines its percentage of seats. The other case corresponds to the situation where the population is divided into L districts; in each district voters vote for one of the two parties and the party getting more vote wins the seat in that district.

There are three principal benchmarks that can be used as reduced forms for the third stage. If there is a single electoral district, we denote by V_k the number (or percentage) of votes obtained by party k. If there are many electoral districts, we denote by V_k^l the number (percentage) of votes obtained by party k in district l.

With one electoral district there are two benchmarks. The first one consists in assuming that the party with the highest V_k implements the policy $\widetilde{a}_k$ announced in stage 1. The second one consists instead in assuming that the policy $\widetilde{a}_k$ is implemented with probability V_k. With L electoral districts, there is a additional benchmark : the party with the highest V_k^l in the highest number of districts implements the policy $\widetilde{a}_k$ announced in stage 1.

Under the presumption that the benefits that these Downsian parties can derive are purely correlated to the event of being in charge of the policy, the objective of a party is easy to define for each of the three benchmarks. In the first benchmark, the payoff of a party k is 1 if k wins the election and 0 otherwise. In the second benchmark, the payoff of party k is V_k. Finally in the third benchmark, the payoff of party k is 1 if party k wins the election in a majority of districts and 0 otherwise.

The behavior of the voters is also very simple when $K = 2$. For the three benchmarks, sincere voting is a dominant strategy. It is therefore easy for the two competitors to predict the outcome of the election as a response to policy announcements up to the residual uncertainty about the types of the voters. In what follows we will assume that the parties are risk neutral. So in the first benchmark, parties maximize the probability of winning the election. In the second benchmark, parties maximize their expected pluralities. In the third benchmark, they maximize the probability of winning a majority of districts.

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We conclude this paragraph by some few comments about the uncertainty resulting from the fact that parties may not know for sure the types of the voters. To illustrate that idea, consider the case where the political type of a citizen is a vector $t_p = (b_1, b_2, c) \in \Re^3$ where b_1 denotes the political preference of this citizen for the ideology of party 1, b_2 denotes his preference for the ideology of party 2 and c represents a signal about the relative "superiority" of party 1 common to all citizens. For the first and second benchmarks, electoral outcomes E can be merged into two : either party1 or party 2 wins the election; we assume that the preferences of the voters are as follows :

$$u(v(t_e, a), E, t_p) = \begin{cases} v(t_e, a) + b_1 + c \text{ if party 1 wins the election and policy } a \text{ is implemented} \\ v(t_e, a) + b_2 \text{ if party 2 wins the election and policy } a \text{ is implemented} \end{cases}$$

Therefore, given the profile of announcements $\widetilde{a} \equiv (\widetilde{a}_1, \widetilde{a}_2)$, a voter with economic type t_e votes for party 1 if :

$$(b_1 - b_2) + c \; > v(t_e, \widetilde{a}_2) - v(t_e, \widetilde{a}_1)$$

If c is not random, the idiosyncratic random variables $b_1 - b_2$ play no role if there is a continuum of voters as the party knows for sure the percentage of citizens of type t_e voting for him. However if c is random, no party can assess with certainty what will be the fraction of votes that it will receive in this subpopulation of citizens, even with a continuum.

In any case, the expected fraction of the population of citizens with economic type t_e voting for party1 is :

$$\pi_1(E(\widetilde{a}_1, \widetilde{a}_2) \mid t_e) \equiv G(\left\{(b_1, b_2, c) \in \Re^3 : (b_1 - b_2) + c \; > v(t_e, \widetilde{a}_2) - v(t_e, \widetilde{a}_1)\right\} \mid t_e)$$

This model is one variant, among many, of the *general model of probabilistic voting.* In this literature, a citizen is non probabilistic if his vote is purely driven by his economic interest i.e. if :

$$u(v(t_e, a), E, t_p) = v(t_e, a)$$

Things are as if a voter was deviating randomly from his economic interest. This model will be used in our manuscript.

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Discontinuous Two-Player Zero-Sum Games

Once we have solved backward the two last stages, we are left with a game between the two parties. For the three benchmarks, this game is a two-player zero-sum game. When A is finite, we can use the all machinery of finite two-player zero-sum games to study the equilibria which of course exist. Note that this game has however no equilibria in pure strategies unless there exists a policy a such that $\pi_1(E(a, a')) \geq \frac{1}{2}$ or $\pi_1(E(a', a)) \leq \frac{1}{2}$for all $a' \in A$ where :

$$\pi_1(E(a, a')) \equiv \int_{T_e} \pi_1(E(a, a') \mid t_e) dF(t_e)$$

is the fraction of voters voting for party 1 when party 1 platform is a and party 2 platform is a'. In the "deterministic" case i.e. when $G(\mid t_e)$ is the Dirac mass in 0, the game is symmetric and an equilibrium exists if and only if there is a Condorcet alternative i.e. an alternative a such that :

$$F(\{t_e \in T_e : v(t_e, a) - v(t_e, a') > 0\}) + \frac{1}{2} F(\{t_e \in T_e : v(t_e, a) - v(t_e, a') = 0\}) \geq \frac{1}{2} \text{ for all } a' \in A$$

As it is well known , such alternative is unlikely to exist unless A is unidimensional and utility functions v are single peaked. In the probabilistic case, existence can be obtained even in mulitidimensional policy spaces but under some strong conditions on $G(\mid t_e)$. General conditions of existence are not known. Laussel and Le Breton (2002) have pointed out that, even in the favorable unidimensional case, if $G(\mid t_e)$ is close enough to, but different from, the Dirac mass in 0, then the probabilistic Downsian game described above fails to possess an equilibrium in pure strategies.

When A is finite, the game has equilibria in *mixed strategies*; it is surprisingly unique under fairly general conditions as demonstrated by Laffond, Laslier and Le Breton (1993). A decent knowledge is available on the *support of the mixed strategy* played at equilibrium; in particular *bounds* on this support are well known.

When A is not finite, things get complicated, as existence of an equilibrium in mixed strategies is not guaranted. This follows from the fact that the Downsian game is very discontinuous : there is a discontinuity at any profile $(\widetilde{a}_1, \widetilde{a}_2)$ where the electorate is divided in two equal parts. No general result of *existence* or *counterexample* is available. Instead, bounds on the support of the mixed equilibria, when they exist, have been provided in McKelvey (1986) and Banks, Duggan and Le Breton (2002).

To conclude that technical but important point, note that payoffs behave a bit better in the case of the second benchmark as in that case the fraction of the population voting for a party varies continuously with respect to $\widetilde{a}$ whenever F has no atoms. If instead, F has atoms, the existence issue is back.

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Overview

A. Example 1 with deterministic voting (Arbitrary K) : the incentives to cultivate favored minorities under alternative electoral systems

B. Example 2 with deterministic voting ($K = 2$): bounds on inequality and the Lorenz curve at equilibrium

C. Example 2 with probabilistic voting ($K = 2$) : partisanship, swing voters and the characteristics of the favored groups

D. Example 4 with deterministic voting ($K = 2$) : inefficient targeting and the impact of the three benchmarks of stage 3 : Winner-Take-All, Proportional System and the implications of the Electoral College

E. Empirical Evidence ($K = 2$)

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The incentives to cultivate favored minorities under alternative electoral systems

We first want to understand if the electoral system itself (independentely of any other consideration) is responsible for some of the inequality in the equilibrium political platforms. When there are at least $K \geq 3$ candidates (parties) competing, the choice of the electoral system is important and we may ask if these electoral systems differ in terms of the incentives that they provide to the candidate to act even in the direction of inequality. This question has been largely unexplored except for an important contribution by Myerson (1993) that we now expose.

The setting which is considered is the setting described in example 1 with λ being the Lebesgue measure on $[0, 1]$: pure redistributive politics in the context of a *continuum* of voters whose voting behavior is purely driven by their economic interest. We also limit our attention for the moment to the case where there is one seat to be allocated and we use the firt benchmark for the third stage.

Let us start with the case where $K = 2$. It is very simple to see here why Nash equilibria in pure strategies fail to exist. Consider any political platform a by party 1. Then, in announcing any platform a' defined as follows

$$a'(t) = \begin{cases} 0 \text{ if } t \in S \\ a(t) + \frac{\int_S a(t)dt}{1-\lambda(S)} \text{ if t} \notin S \end{cases}$$

where $S \subset [0, 1]$ is any subset such that $\lambda(S) < \frac{1}{2}$ and $\int_S a(t)dt > 0$, party 2 is certain to win the election. So, we have to look for Nash equilibria in mixed strategies. A mixed strategy in this game could be in principle a very complicated object since the space of pure strategies is large. Here, we discuss the case where the offers of transfers made by candidate $k \in \{1, 2\}$ to voters are realizations of random variables having the same cumulative distribution function $H_k : \Re \to [0, 1]$. Voters are treated alike ex ante but will be different ex post. Because there is a continuum of voters, H_k will be the empirical distribution of offers in the electorate : $H_k(x)$ is the fraction of citizens in the electorate that receive promises below x by candidate k. The set of "mixed" strategies of both candidates is here the set of cumulative distribution functions H satisfying the feasibility constraint :

$$\int_0^\infty x dH(x) = 1$$

Given a profile (H_1, H_2) of mixed strategies, the fraction of citizens $\pi_1(H_1, H_2)$ voting for party 1 is :

$$\pi_1(H_1, H_2) = \int_0^\infty H_2(x) dH_1(x)$$

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Candidates do not face any uncertainty : the outcome of the election is certain for any pair (H_1, H_2). In looking for equilibria, we can equivalently consider $\pi_k(H_1, H_2)$ or $f(\pi_k(H_1, H_2))$ (where f is the discontinuous function depicted on figure 1) to be the payoff of candidate k.

Insert Figure 1 here

In addition to be zero-sum, note that the game is symmetric. In an equilibrium, both candidates must get expected payoffs of 0. If we denote by H the offer distribution of both candidates in a symmetric equilibrium, it must therefore be solution of the equation :

$$\int_0^\infty H(x) dH(x) = \frac{1}{2}$$

Myerson has proved that there is a unique symmetric equilibrium : each candidate makes offers from the uniform distribution over the interval from 0 *to* 2; that is $H(x) = \frac{x}{2}$ if $x \in [0, 2]$.

To evaluate the ex post unequal character of the offer, it is useful to calculate the Lorenz curve L induced by H. By definition for an arbitrary distribution H on $[0, +\infty[$with a finite first moment :

$$L(t) = \frac{\int_0^{H^{-1}(t)} x dH(x)}{\int_0^\infty x dH(x)} \text{ for all } t \in [0, 1]$$

$L(t)$ represents the fraction of the "cake" obtained by the $t\%$ poorest fraction of the population. Of course, by construction $L(t) \leq t$. It is straightforward to check that the Lorenz curve L associated to Myerson's equilibrium is :

$$L(t) = t^2$$

The poorest half of the population gets one quarter of the total; the first decile gets 1% of the total.

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We now move to the analysis of multicandidate competition with the rank-scoring rules defined earlier. We are also going to focus on symmetric equilibria where candidates select offer distributions as in the case where $K = 2$. Myerson assume that voters are going to vote sincerely : he argues that this is a consequence of the symmetry assumption as in the event that his vote could actually decide a close race between two of the candidates, it is equally likely to be any pair of the K candidates who are in a close race. Thus, each voter should rank the candidates in order of their offers, giving s_1 points to the candidate who offer the most, s_2 points to the candidate who offer the second-most, and so on.

Let us now perform the equilibrium anlaysis from the point of view of candidate k given that all other candidates use the offer distribution H. When candidate k offers x to a voter, the probability that candidate k is ranked in position j by this voter is $P(j, H(x))$ where we let:

$$P(j,q) \equiv q^{K-j}(1-q)^{j-1}\frac{(K-1)!}{(K-j)!(j-1)!}$$

Then, the expected value of the points that this voter will give to candidate k is $R(H(x))$, where we let :

$$R(q) = \sum_{j=1}^{K} P(j,q)s_j$$

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In a symmetric equilibrium, all candidates must get the same expected score, which must equal $\overline{s}$. Thus, there is an equilibrium in which all candidates use the cumulative distribution H if and only if :

$$\int_0^\infty R(H(x))dG(x) \leq \overline{s} \text{ for any cumulative distribution } G \text{ on } \Re_+ \text{ such that } \int_0^\infty xdG(x) = 1$$

Remarkably, Myerson has proved that there is a unique such H. Its support is the interval $\left[0, \frac{1}{\overline{s}}\right]$ and it satisfies the equation :

$$x = \frac{R(H(x))}{\overline{s}} \text{ on } \left[0, \frac{1}{\overline{s}}\right]$$

With this result, we are in position to discuss the impact of the scoring vector $(s_1, s_2, ..., s_K)$ on the offer distribution H. First we see that the maximal offer is the reciprocal of the average of the ranking points. Since $\overline{s}$ is smallest under plurality voting, there is a sense in which we can say that this electoral system encourages inequality. For plurality voting, $R(q) = P(1, q) = q^{K-1}$. So the equilibrium cumulative distribution satisfies :

$$x = K(H(x))^{K-1} \text{ for all } x \in [0, K]$$

and so :

$$H(x) = (\frac{x}{K})^{\frac{1}{K-1}} \text{ for all } x \in [0, K]$$

When $K = 4$, each candidate offers less than $\frac{1}{2}$ to half of the voters, less than 1 to 65% of the voters, but makes offers more than 2 to 20% of the voters. When $K = 10$, each candidate offers less than $\frac{2}{100}$ to half of the voters !

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Among rank-scoring rules, the lowest maximal offer is $\frac{K}{K-1}$ which is achieved by negative plurality voting. In this sense, negative-plurality voting gives us the most egalitarian solution. No voter is offered more than 1.34 when $K = 4$. The equation describing H is here :

$$x = K\frac{(1 - (1 - H(x))^{K-1}}{K - 1}$$

This formula implies, for example, that each candidate offers less than 1 to only 37% of the voters when $K = 4$ and less than 1 to only 23% of the voters when $K = 10$.

The value of the standard deviation of H for all scoring rules where each voter must distribute V noncumulative votes is reported on table 1 below.

Insert Table 1 here

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Borda voting is another important rank-scoring rule. We can show that for that rule,the equilibrium offer distribution is the uniform distribution over $[0, 2]$.

Myerson's paper contains much more information. He provides also an examination of approval voting and single transferable vote. The explicit calculation of the equilibrium offer distributions is difficult. Myerson analyses approximate solutions for a discrete version of the problem. In each case, he found only one equilibrium. Some results are reported in table 2 below.

Insert Table 2 here

Maximal offers are between 1 and 2. The support of H is not convex when $K \geq 3$. Each candidate gives offers that are greater than 1 to a majority of the voters.

Myerson's paper contains also an anlysis of multiseat elections. He notes in particular that in the equilibrium for the second benchmark of stage 3 is exactly the equilibrium distribution derived for the plurality-scoring rule. This is not surprising : we have already pointed out that uncertainty disappears here and that maximizing the probability of winning or the size of the electoral support leads to the same equilibria.

Myerson's paper provides also a general discussion of minority representation and a comparison of his analysis with Cox (1987, 1990) analysis of centripetal and centrifugal incentives in electoral systems. He found that electoral sytems that according to Cox encourage more diversity of candidate positions generally also incite candidates to create more inequality among voters.

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Bounds on inequality and the Lorenz curve with a Finite Number of Groups

Myerson's analysis of example 1 uses a continuum of voters : this assumption was essential to make meaningful the fact that each candidate could make independent offers to the different voters. We now turn our attention to example 2 with $M = 1$ and an equal number of citizens in each group i.e.

$$A = \left\{ a \in \Re_+^J : \sum_{j=1}^{J} a^j = 1 \right\}$$

Further, from now on, the analysis is limited to the case where $K = 2$. In that case, the offers to the J groups cannot be independent.The symmetric two-player zero-sum game belongs to the family of "Colonel Blotto" games analysed first by Borel (1936) and Gross and Wagner(1950). We observe that in this setting the games where candidates maximize respectively their probability of winning and their electoral suppport are not equivalent; an example in Laffond, Laslier and Le Breton (1994) even shows that the supports of the two mixed Nash equilibria may be disjoint for an arbitrary A. A mixed strategy is here a probability distribution over the unitary simplex of $\Re^J$. Laslier and Picard (2002) provide a thorough analysis of the equilibria of the discontinuous game where candidates maximize their electoral support. Let p be any probability distribution on A. They show that if the J marginals of p are uniform on $\left[0, \frac{2}{J}\right]$, then p is an equilibrium strategy but unlike Myerson, they do not prove that any optimal strategy must satisfy this uniformity property on marginals.

Given this preliminary step, we are left with the following mathematical problem : construct a J dimensional random vector p such that :

$$Supp(p) \subseteq A \text{ and each random variable } p_j \text{ is uniform on } \left[0, \frac{2}{J}\right] \text{ for } j = 1, 2, ..., J$$

They point out the existence of multiple solutions to that problem. They however shows that any equilibrium strategy p is bounded as follows in the following sense :

$$Supp(p) \subseteq Hex \equiv \left\{a \in A : a^j \leq \frac{2}{J} \text{ for } j = 1, 2, ..., J\right\}$$

This suggests that the chaos resulting from majority voting in this multidimensional policy space is however bounded as no group can receive more that twice the average endowment. If inequality is descibed by the maximal offer that can be made, then the bound result obtained by Laslier and Picard puts some limits on the favors distributed by the candidates.

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Most of the paper focus on the following geometric solution. Consider the regular J-gone $[P_0, P_1, ..., P_{J-1}]$ defined by the points :

$$P_j = (r_J \cos \frac{(2j-1)\pi}{J}, r_J \sin \frac{(2j-1)\pi}{J}, 0)$$

for $j = 0, ..., J-1$ and

$$r_J = \frac{2}{n\sqrt{1 + \cos \frac{2\pi}{J}}}$$

The largest disk D_J inside this J-gon is centered at the origin and has radius $\frac{1}{J}$. Consider the sphere centered at the origin with radius $\frac{1}{J}$, pick randomly a point M on the surface of the sphere and project this point on the disk D_J. Let $P(M)$ be the projection of M. Then with some geometry, it is possible to show that the random vector a where a^j is the distance between $P(M)$ and the line $P_{j-1}P_j$ is a solution to the above problem. We note that the support of this solution is two dimensional.

Laslier and Picard provide a very detailed analysis of the inequality of the distributions. We will limit our attention here to the expected Lorenz curve. They show that, when we let J goes to infinity while keeping the mean equal to $\frac{1}{J}$ the expected Lorenz curve L induced by the solution described above is given by the formula :

$$L(t) = t - \frac{1}{4} \sin \pi t$$

which differs from Myerson's quadratic. Here, on average, the 20% poorest receive about 5.3% of the total and the 20% richest ones gets about 34.7% of the total. The fact that the limit Lorenz curve in Laslier and Picard 's analysis does not coincide with Myerson's analysis in the limit case of a continuum of voters is intriguing !

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Partisanship, Swing Voters and the Characteristics of the Favored Groups

Until now, we have investigated to which extent electoral systems provide incentives for parties to move away from equal distribution of resources across citizens or groups of citizens in situations where these citizens or groups could not be considered different from an economic point of view. We have studied the pure tactical strategic motive for parties to target a specific group in order to enhance their chances of get access to more political power with all the benefits associated with that. The wo examples that we have examined were symmetric in that all the citizens or groups of citizens were all alike in any respect. Therefore we were not in position to say something about the identity and characteristics of groups receiving a greater share of the favors as the result of electoral competition. We have just been able to evaluate the unequal character of the distributive policies through anonymous statistics answering questions like for instance : how much is obtained by the first decile ? But up to any rearrangement, any citizen could be in the first decile. We now want to understand who is in the first decile and so on.

To proceed, we are going to consider the probabilistic model defined earlier in the context of example 2 with groups of equal size, $T_e = \{1, 2, ..., J\}$ and $v(a, j) = v(a(j), j)$ with some self explanatory abuse in the notation. We also assume that the random variable c is identically equal to 0 and denote by b the relevant random variable $b_1 - b_2$ and by $G(| j)$ its cumulative distribution function. Then the fraction of voters in group j voting for party 1 for the profile $\widetilde{a} \equiv (\widetilde{a}_1, \widetilde{a}_2)$ of policy announcements is :

$$G(v(\widetilde{a}_1(j), j) - v(\widetilde{a}_2(j), j) \mid j)$$

Thus, the total fraction of citizens voting for party 1 in the circumstances $(\tilde{a}_1, \tilde{a}_2)$ is :

$$\sum_{j=1}^{J} G(v(\tilde{a}_1(j), j) - v(\tilde{a}_2(j), j) \mid j)$$

This is the model considered and explored by Dixit and Londregan (1995,1996) and Lindbeck and Weibull (1987) under some set of assumptions on v and $G(\mid j)$.

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First, let us focus on the symmetric case where no party has an initial advantage. This is already reflected in the assumption $c \equiv 0$; we also assume that that $G(\mid j)$ is symmetric around 0 for all $j = 1, ...J$. In fact some of the results below, on top of which the equilibrium convergence of the two programs of the two candidates, hold true even in the asymmetric case. We also assume that $v(., j)$ and $G(. \mid j)$ are twice continuously differentiable.

Under these strong conditions a symmetric, equilibrium in mixed strategies exists as the game is continuous. It is not clear that an equilibrium in pure strategies exists. Under the presumption that there is one, consider the conditions satisfied by an equilibrium platform a must satisfy when no group j is ignored. The first order conditions are described by the following equations :

$$v'(a_j, j) g(0 \mid j) = \lambda \text{ for all } j = 1, ..., J$$

and

$$\sum_{j=1}^{J} a_j = 1$$

where $\lambda \geq 0$ and $g(\mid j)$ denotes the density of $G(\mid j)$. The formula above brings two insights when $v(., j)$ is strictly concave for all $j = 1, ...J$. First, if $v'(, j) = v'(, k)$, then $a_j > a_k$ if and only if $g(0 \mid j) > g(0 \mid k)$. Second, if $g(0 \mid j) = g(0 \mid k)$ and $v'(x, j) > v'(x, k)$ for all x, then $a_j > a_k$.

Let us interpret these two observations. The parameter $g(0 \mid j)$ represents the fraction of voters in group j who are very much willing to switch from party 1 to party 2 and vice versa for a slight differential treatment in policies. Their degree of partisanship is very low and they are call swing voters.The first insight is that everything equal otherwise, groups with more swing voters will receive more of the total resources. The second insight is that the amount of resources obtained by a group will increase with respect to the marginal rate of substitution of the group between private resources and the ideological differential between the two parties : in particular if $v(a_j, j) = \widetilde{v}(a_j + e_j)$, then groups j with a lower initial endowment (i.e. gross income) get more when we controm for other variables. Both insights are quite obvious and it may be more relevant at that stage to understand the magnitude of this effects and the conditions which guarantee that the all equilibrium analysis is correct. Before moving to that, note that it is now an empirical matter to identify which groups are money oriented and having a large fraction of swing voters. Some political scientists have suggested that senior citizens, californians and garment workers are matching these criteria.

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Assume that for all $j = 1, ..., J$: $v(x, j) = x^{\alpha_j}$ with $\alpha_j \in]0, 1[$ and that $G(u \mid j)$ is Gaussian i.e.

$$g(u \mid j) = \frac{1}{\sigma_j \sqrt{2\pi}} e^{-\frac{u^2}{2\sigma_j^2}}$$

Under these parametric assumptions, the equations above simplify to :

$$\begin{aligned} (a_j)^{\alpha_j - 1} &= \frac{\lambda \sigma_j}{\alpha_j} \text{for all } j = 1, ..., J \\ &\text{and} \\ \sum_{j=1}^{J} a_j &= 1 \end{aligned}$$

We deduce then that for all $j, k = 1, ..., J$:

$$a_k = (a_j)^{\frac{\alpha_j - 1}{\alpha_k - 1}} (\frac{\alpha_j \sigma_k}{\alpha_k \sigma_j})^{\frac{1}{\alpha_k - 1}}$$

and therefore a_j is solution of the following algebraic equation in the variable a_j:

$$\sum_{\substack{k=1 \\ k \neq j}}^{J} (a_j)^{\frac{\alpha_j - 1}{\alpha_k - 1}} (\frac{\alpha_j \sigma_k}{\alpha_k \sigma_j})^{\frac{1}{\alpha_k - 1}} = 1 - a_j$$

An explicit solution is not immediate in the general case. In the case where the marginal utility of money is the same across groups i.e. $\alpha_j \equiv \alpha$ for all $j = 1, ..., J$, we obtain :

$$a_j = \frac{(\sigma_j)^{\frac{1}{\alpha - 1}}}{\sum_{k=1}^{J} (\sigma_k)^{\frac{1}{\alpha - 1}}} \text{ for all } j = 1, ..., J$$

When α is close to 0, then the formula says that each group gets a fraction proportional to reciprocal the standard deviation but when $\alpha = \frac{1}{2}$, the relevant characteristic of the group is the reciprocal of the variance. In any case in this Gaussian setting, the lower is the dispersion in a group, the larger is the fraction of total resources received by that group.

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Note also that if the marginal utility of income is constant, then the equilibrium cannot be interior. It is straightforward to see that in such case, only the groups with the largest fraction of swing voters get something i.e.

$$a_j > 0 \text{ iff } g(0 \mid j) \geq g(0 \mid k) \text{ for all } k = 1, ..., J$$

Another important issue is the scope of validity of this first order approach resulting from the assumption that an equilibrium in pure strategies exist. From classical results in game theory, existence will be guaranted if the payoff of each of the candidate is concave in its own strategy. Here this amounts to :

$$v''(\tilde{a}_1(j), j)g(v(\tilde{a}_1(j), j) - v(\tilde{a}_2(j), j) \mid j) + (v'(\tilde{a}_1(j), j))^2 g'(v(\tilde{a}_1(j), j) - v(\tilde{a}_2(j), j) \mid j) \leq 0 \text{ for all } j = 1, ..., J$$

As noted by Lindbeck and Weibull, a condition for this to be true is :

$$\underset{x \in \Re_+}{Inf} \frac{\mid v"(x, j) \mid}{v'(x, j)^2} \geq \underset{u \in \Re}{Sup} \frac{\mid g'(u \mid j) \mid}{g(u \mid j)}$$

This is a very demanding sufficient condition involving a comparison of some concavity index of the function $v(., j)$ and a characteristic of the probability distribution $G(\mid j)$. Note that for the logarithmic function $v(x, j) = Log(x + \gamma_j)$ where γ_j is a constant, the left hand side of the above inequality is equal to 1. In the Gaussian case, the right hand side is equal to $\underset{u \in \Re}{Sup} \frac{u}{\sigma_j^2} = +\infty$ and the condition is therefore violated. If instead, $G(u \mid j) = \frac{e^{\sigma_j u}}{1+e^{\sigma_j u}}$ where σ_j is a positive parameter, then it is straightforward to verify that $\underset{u \in \Re}{Sup} \frac{|g'(u|j)|}{g(u|j)} = \sigma_j$. In that case, the Lindbeck-Weibull sufficient condition is satisfied iff $\sigma_j \leq 1$. Note that when $\sigma_j \to +\infty$, $G(\mid j)$ converges weakly to the Dirac mass in 0 i.e. to the deterministic case and that here $\sigma_j = g(0 \mid j)$. The lesson is that if there are too many swing voters the condition is unlikely to be satisfied. As the condition is sufficient, we cannot conclude however concerning the existence of an equilibrium.

Banks and Duggan (1999), Laussel and Le Breton (2002) but also Van der Straeten (2000) for the specific case of the Coughlin's redistribution model have noted that the Downsian game fails to have an equilibrium in pure strategies if, in each group, there is a sufficiently large fraction of swing voters. This is bad knews for all those who were using probabilistic voting as a device to rescue existence. It will depend on some key electoral parameters namely the proportion of swing voters in each group. This was noted a long ago by Kramer (1978) in reaction to Hinich (1977,1978).

Practically, this means that we will have to consider Nash equilibria in mixed strategies as we did until now to see whether some of the main claims resists to the equilibrium analysis when there are many swing voters among the citizens. This is one of the main objective of the new model presented in the second part of this paper.

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Winner-Take-All versus Proportional Electoral Systems

To present the last theme of this overview, we come back to the deterministic setting that we have used prior to the presentation of the third theme : here all the voters are swing voters. The setting that we consider is due to Lizzeri and Persico (2001) and has been presented as example 4 in this paper. Precisely, we assume that λ is the Lebesgue measure, $T_e = N = [0,1]$ and that $e(n) = 1$ for all n. Then :

$$A = \left\{ \begin{array}{c} a = (g,t) \text{ where } g \in \{0,1\} \text{ and } t : N \to \Re_+ \text{ are such that } : g + \int_0^1 t(n)dn = 0 \\ \text{and } a(n) \geq -1 \text{ for all } n \in N \end{array} \right\}$$

We have already noted many times that with a continuum of voters, the Nash equilibria of the Downsian game is invariant to the particular choice of payoff of the candidates as long as this payoff is not random and a weakly monotonic function of the electoral support of the candidate. An immediate implication of that observation is that for such settings, a winner-take-all electoral system leads to the same outcome as a proportional electoral system.

To analyse the impact of the choice between these two benchmarks for stage 3, we either need to abandon the idea of a continuum of voters without common values as in Persson and Tabellini (1999, 2000), or to go for a setting like example 2 as in Lindbek and Weibull or to enlarge the space of policies as in example 4. The first direction leads to conclusions similar to those of Lizzeri and Persico and will not be presented here. Lindbeck and Weibull provide some partial but interesting hints along the direction but report as we did on the difficulty of deriving general results. We now move straight to the analysis of example 4 which is the simplest framework to discuss these issues. In their framework uncertainty is introduced via the provision of public good : with a mixed strategy of providing the good a common value component is introduced in the voting behavior and now each candidate faces strategic uncertainty. We denote by θ the value of the public good for all citizens; we assume that θ is common knowledge.

In comparison to example 1, the two candidates have now an additional strategic instrument namely g. What is the value for them of that instrument ? There is an obvious tradeoff. On one hand, if $\theta > 1$, selecting that g is a way to please everybody as compared to the status quo situation where each citizen is left with his endowment. On the other hand, by definition of a public good this is an egalitarian policy which prevents candidates to target some specific groups in order to gain their electoral support.

Here a mixed strategy for a party is a pair (q, H) where $q \in [0, 1]$ and H is an offer cumulative distribution as already defined in presenting Myerson's analysis. Here q is the probability that the candidate select g as his platform; so a policy selects g with probability q or to make offers according to H with probability $1 - q$. Note that here if θ is large enough the Downsian games corresponding to benchmarks 1 and 2 have an equilibrium in pure strategies. The policy g can be defeated iff $\theta < 2$; if $\theta \geq 2$ both candidates announce g.

Consider first the first benchmark : the party winning the election takes all. In that case, both candidates maximize their probabilities of winning the election. In that case, Lizzeri and Persico prove that if $\theta \in]1, 2[$, then the Downsian game has a unique equilibrium (q, H) with $q = \frac{1}{2}$ and :

$$H(x) = \begin{cases} 0 \text{ if } x \in]-\infty, -1] \\ \frac{1}{2}(\frac{x+1}{2-\theta}) \text{ if } x \in [-1, 1-\theta] \\ \frac{1}{2} \text{ if } x \in [1-\theta, \theta - 1] \\ \frac{1}{2}(1 + \frac{x+1-\theta}{2-\theta}) \text{ if } x \in [\theta - 1, 1] \\ 1 \text{ if } x \in [1, +\infty[\end{cases}$$

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Consider now the second benchmark. They demonstrate that if $\theta \in]1, 2[$, then this new Downsian game has also a unique equilibrium (q, H) with $q = \theta - 1$ and H as in the first benchmark.

The redistribution of money is the same in both electoral systems. It is interesting to observe that the support of H is not convex. Precisely, there are two symmetric groups at equilibrium as illustrated on figure 2 below.

Insert figure 2 here

On the other hand, the level of public good provision is higher with the proportional electoral system. The intuition is as follows. When a candidate evaluates a deviation with respect to H, offering more than $\theta - 1$ to more than 50% of the voters is profitable if the opponent offers to provide the public good but not when he redistributes. When the opponent redistributes, the best deviation is to downgrade the offers of $\theta - 1$ down to offers of $1 - \theta$, since no voter is receiving offers in the interval $[1 - \theta, \theta - 1]$. The vote share gained by this deviation increases with θ, since the higher θ, the greater the savings. In the proportional system the strength of this deviation increases with θ and so at equilibrium, to discourage this deviation, the probability that the opponent offers the public good must be high enough and increasing with θ. By contrast, in the winner-take all system the value of this deviation is independent of θ as the margin of victory is irrelevant. In fact the value of the opposite deviation, offering more than $\theta - 1$ to more than 50% of the voters is also relevant and independent of θ.

We are now in position to compare the two electoral systems. Their outcomes differ when $\theta \in]1, 2[$ but are both *ex post Pareto efficient*. We focus on the respective *ex ante efficiencies* of the two systems. Here each voter is confronted to a randomization since candidates uses mixed strategies but voters are treated similarly. It is obvious to see that both systems are ex ante inefficient since any voter would prefer to produce the public good for sure. It is also straightforward to see that, *if the voters are risk averse or risk neutral, the proportional system is superior to the winner-take-all system from the point of view of ex ante Pareto efficiency iff* $\theta > \frac{3}{2}$.

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We conclude this analysis by an examination of the third benchmark : the country is divided into districts and one representative is elected in each district. Lizzeri and Persicoassume that there is a continuum of districts identical both in size and in the benefit they receive from the public good. A strategy must now specify, in the case where redistribution is chosen, the aggregate transfer to each district as well as the distribution inside the district. This last qualification shows that this setting differs from example 2 as no district measurability constraints are imposed on transfers.

The electoral college system is going to be more inefficient than the the nationwide winner-take-all sytem since a candidate who offer to provide the public good must now worry about his opponent going after a majority of voters in a majority of districts i.e. after about 25% of the voters. If $\theta < 4$, a candidate can offer more than $\theta - 1$ to more than 50% of the voters in more than 50% of the states. This strategy leads to a sure victory against a candidate offering the public good with probability 1. Lizzeri and Persico has proved that when $\theta \in [1, 4]$, the equilibrium probability of providing the public good is less than $\frac{1}{2}$. They also prove that when $\theta < 1$, the public good is never provided and the offer distribution H is made according to the following process : each candidate k draws the aggregate transfer μ_k to any district from a uniform distribution on $[-1, 1]$ and in each district candidate k makes offers according to a uniform distribution on $[-1, 2\mu_k + 1]$.

Here, it cannot be an equilibrium that all districts get the same resources as a candidate could deviate by targeting higher average offers to a majority of districts. This there must be ex post differences in the amount of resources obtained by the different districts.

We have already pointed out that *the electoral college is more ex ante inefficient that the nationwide winner-take-all system in terms of public good provision.* Further it leads to less egalitarian distribution of offers. The maximal offer is now 4 instead of 2 and it can be shown that the nationwide system is prefered to the electoral college by any risk averse citizen.

This conclusion has been derived under the assumption that offers could differ within the same district which is quite debatable. If this not the case, the situation is analogous to the nationwide situation, at least with a continuum of districts. Still, it remains that a sort of some pork-barrel politics will be at work here. Note also that we have assumed here that the objective of each party was to win the national election; this does not account for the fact that candidates in each district may be themselves interested in winning in their district and pay therefore attention to the promises made to their district during the electoral campaign. If this agency conflict between the party and its local candidates was serious, then the above analysis should revisited. We would then have to develop a model to explain how candidates are nominated and possibly monitored, a topic which is far beyond the scope of the current manuscript.

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We are now in position to motivate the introduction of our new and simple model. First, let us mention that we will limit our investigation to the case where $K = 2$ and the second benchmark i.e. parties aim to maximize their electoral support. Given that, we dont want to assume a continuum of voters or more precisely we dont want to assume the absence of measurability constraints. The number of characteristics on which a politician can base a promise is limited and it seems reasonable that except some few circumstances, it is not based on the name of a person. Therefore, we are interested in the setting described as example 2 already analysed by Laslier and Picard. But we want also to consider some political heterogeneity among groups along the lines analysed by Dixit and Londregan and Lindbeck and Weibull in order to understand which groups are likely to get more favors as the result of this Downsian electoral competition. However, we have seen that their analysis relying on the existence of an equilibrium in pure strategies was not valid whenever there were too many swing voters. A unified analysis should therefore be based on Nash equilibria in mixed strategies.

The rest of the paper is devoted to that task. We introduce a model along the lines of example 2 enriched by some probabilistic voting. Then we proceed to the equilibrium analysis of that model and compare our results with early contributions.

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A Simple Model

The setting is as described in example 2 i.e. the population is partitioned into finitely many groups $\left\{N^1, N^2, ..., N^J\right\}$and

$$A = \left\{ \begin{array}{c} a : T_e \longrightarrow \Re_+ \text{ such that } : \int_N a(t_e) d\lambda(t_e) = M \\ \text{and } a(t_e) = a(t'_e) \text{ for all } t_e, t'_e \in N^j \text{ for some } k = 1, .., J \end{array} \right\}$$

In that case, the set A can alternatively be described as :

$$A = \left\{ a \in \Re^J_+ : \sum_{j=1}^{J} n^j a^j = M \right\}$$

where $n^j \equiv \lambda(N^j)$ for all $j = 1, ..., J$. We assume without loss of generality that $v(a, t_e) = \widetilde{v}(a(t_e))$ for all $t_e \in T$ with $\widetilde{v}$ increasing. We depart from Laslier and Picard by assuming that in each group there are three types of voters i.e. $T_p = \{1, 2, S\}$ with the following interpretation. A voter with political type $t_p = k$ for $k = 1, 2$ always vote for candidate (party) k regardless of their "economic" utility differential between the two platforms. This form of extreme partisanship can be associated with a strong preference for one party's fixed positions over those of the other party : as formulated by Grossman and Helpman (2001)"By partisan, we mean a voter who ardently favors a certain political party". In contrast, a voter with political type $t_p = S$ is a pure swing voter : he always vote for the party announcing the policy matching better his economic interest and is pays no attention to the differences on fixed positions between the parties. This discrete set of political types is of course an approximation but rich enough to bring some important insights on the impact of the political characteristics of groups on what they get out of the electoral competition. Groups differ in terms of the fractions of partisans for the two parties and swing voters. Let

n_1^j, n_2^j and n_S^j denote respectively the number of voters in group j with political type $1, 2$ and S.who will, in any case, vote for parties A and B. Of course :

$$n_i = n_1^j + n_2^j + n_S^j.$$

We denote by θ^j the fraction of swing voters in group j, among all the swing voters i.e. :

$$\theta^j = \frac{n_S^j}{\sum_{j=1}^J n_S^j}.$$

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If party 1 proposes $a \in A$ and party 2 proposes $b \in A$, the net plurality in favor of 1 is:

$$\sum_{j=1}^{J} (n_1^j - n_2^J + n_S^j sign\left[a_j - b_j\right])$$

Obviously the two-player zero-sum game where both parties maximize their net pluralities is equivalent to the symmetric two-player zero-sum game where the payoff of party 1 is :

$$\pi(a, b) \equiv \sum_{j=1}^{J} \theta^j sign\left[a_j - b_j\right]$$

where, for any $t \in \Re$, $sign(t)$ equals -1 if $t < 0$, 0 if $t = 0$ and $+1$ if $t > 0$. A mixed strategy is a regular probability distribution on A; if p and q are two mixed strategies the payoff to p against q is:

$$\pi(p, q) = \int_{a \in A} \int_{b \in A} \pi(a, b) dp(a)) dq(b)$$

Since the game is symmetric, the value of this game, under the presumption that it is well defined, is equal to 0. A mixed strategy p^* is *optimal* if

$$\pi(p^*, b) \geq 0 \text{ for all } b \in A$$

A profile of mixed strategies (p^*, q^*) is a Nash equilibrium iff both p^* and q^* are optimal. It is simple to see that this Downsian game has a Nash equilibrium in pure strategies iff $\underset{1 \leq j \leq J}{Max} \theta^j \geq \frac{1}{2}$. When $\underset{1 \leq j \leq J}{Max} \theta^j > \frac{1}{2}$, the unique optimal strategy is to allocate the all budget M to group j. If $\underset{1 \leq j \leq J}{Max} \theta^j = \frac{1}{2}$ and $J \geq 3$, then the unique optimal strategy is still to allocate the all budget M to group j. If $\underset{1 \leq j \leq J}{Max} \theta^j = \frac{1}{2}$ and $J = 2$, then any strategy is optimal. In what follows, we assume, without loss of generality, that $J \geq 3$ and $\frac{1}{2} > \theta^1 \geq \theta^2 \geq .... \geq \theta^J$.

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Since the game is discontinuous, we cannot appeal to the classical theorems in game theory textbooks to assert the existence of Nash equilibrium in mixed strategies. The existence issue in discontinuous games has been investigated in many recent papers including Dasgupta and Maskin (1986), Mertens (1986) and Reny (1999) to cite a few. One direction could consist in verifying that at least one of the sufficient conditions discovered by these authors holds in our setting. For instance, Artale and Gruner (2000) prove existence of an equilibrium in mixed strategies in the case where $J = 3$ and $n_S^j = n^j = \frac{1}{3}$ for all $j = 1, 2, 3$ by using a condition due to Dasgupta and Maskin. Duggan (2002) contributes also to this research agenda for the related but different Downsian model with spatial preferences. Here, existence follows from the fact that we construct an optimal strategy; the added value of this proof is to offer the possibility of investigating the property of the equilibrium, in particular the potential unequal character of the distribution of M among groups.

Before moving to that, we first examine the possible bounds that can be placed on the distributions played with some positive probability in an optimal strategy. If these bounds are not too degenerate, then they suggest that there is a limit to the chaos resulting from the multidimensional character of the policy space.

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The Uncovered Set

The concept of uncovered set introduced below is an application of the general concept of weighted uncovered set introduced and discussed in De Donder, Le Breton and Truchon (2000) and Dutta and Laslier (1999). The covering relation is more demanding than the traditional weak dominance relation; some of the relations between the two notions were first analysed in Mc Kelvey (1986) but the first definition of the uncovered set (when weigths are equal) is due to Fishburn (1977) and Miller (1980). Banks, Duggan and Le Breton (2002) have proved that the support of any mixed strategy in a class of games including the game analysed here is a subset of UC.

Let $a, b \in A$. We say that a covers b if $\pi(a, b) > 0$ and $\pi(a, c) \geq \pi(b, c)$ for all $c \in A$. The uncovered set, denoted UC hereafter, is the set of policies in A which are not covered.

The computation of UC proceeds through a sequence of steps. Let V^j be the policy allocating the all budget M to group j.

Step 1. For any $a, b \in A' \equiv A \backslash \bigcup_{1 \leq j \leq J} \{V^j\}$ with $a \neq b$, there exist $c \in A'$ such that $\pi(a, c) > \pi(b, c)$. Further, if $\pi(a, b) \neq \theta^J - \sum_{j=2}^{J-1}$, then c can be chosen arbitrary close to b.

Step 2. $UC = A$

This result echoes Laslier and Picard and Epstein (1998) who demonstrated a somewhat similar chaos result for the Downsian game where parties maximize their probability of winning the election.

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The Minimal Covering Set

We now move to a refinement of the uncovered set which has been proposed first by Dutta (1988) as a social choice solution. A positive game theoretical foundation of this set was provided by Duggan and Le Breton (1996) when the majority has no ties. Generalizations of the minimal covering set for weighted situations were provided by De Donder, Le Breton and Truchon and Dutta and Laslier.

Let $a, b \in A$ and $B \subseteq A$. We say that a covers b in B if $\pi(a,b) > 0$ and $\pi(a,c) \geq \pi(b,c)$ for all $c \in B$. The subset B is a covering set if for any $b \in A \backslash B$, there exists $a \in B$ such that a covers b in B.

When A is finite, one can prove that there exists a unique smallest (with respect to inclusion) covering set; this set is called the **minimal covering set**. Here, it is nevertheless possible to exhibit a proper subset of A which is almost a covering set.

Step 3. Let $Hex(\theta) \equiv \left\{ a \in A : a^j \leq \frac{2\theta^j}{n^j} M \text{ for all } j = 1, 2, ..., J \right\}$. Then for any $b \in A \backslash Hex(\theta)$, there exists $a \in Hex(\theta)$ such that a covers b in the relative interior of $Hex(\theta)$.

In the fnite case, it is well known that the support of any mixed Nash equilibrium is a subset of the minimal covering set. We will see that here the set $Hex(\theta)$ is also a bound to the support of any optimal play. This shows that too unequal distributions, in the sense of giving too much to a single group, cannot be played by the two parties in equilibrium.

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We now move to the construction of *an* optimal play. The key step is the following.

Step 4. Let p be a mixed strategy such that the probability law of the i^{th} marginal is the uniform distribution on $\left[0, \frac{2\theta^j}{n^j} M\right]$. Then, p is an optimal strategy.

We have now a well defined problem : how to construct a J-dimensional random vector with support in A and with marginals having the distributions described in step 4. The analysis follows closely Laslier and Picard and Laslier (2002).

Step 5. It is possible to merge the groups

(i) either into three sets S^k $k = 1, 2, 3$ such that $\beta^k \equiv \sum_{j \in S^k} \theta^j < \frac{1}{2}$ for all $k = 1, 2, 3$.

(ii) or into four sets S^k $k = 1, 2, 3, 4$ such that $\beta^k \equiv \sum_{j \in S^k} \theta^j = \frac{1}{4}$ for all $k = 1, 2, 3, 4$.

For the time being, we focus on case (i) in step 5. Case (ii)will be analysed separately. We now define a three-dimensional random vector $(\alpha^1, \alpha^2, \alpha^3)$.

Consider the triangle $A^1A^2A^3$ with sides A^2A^3, A^1A^3 and A^2A^3 of respective length β^1, β^2 and β^3. Let Ω be the incenter of $A^1A^2A^3$, r be the radius of the circle with center Ω inscribed inside $A^1A^2A^3$ and S be the sphere with center Ω and radius r. Let M be a point randomly chosen on S according to the uniform distribution on S and let $M\prime$ be the projection of M on the plane containing $A^1A^2A^3$. Define α^1 to be the distance from $M\prime$ to A^2A^3, α^2 to be the distance from $M\prime$ to A^1A^3 and α^3 to be the distance from $M\prime$ to A^1A^2.

Step 6. For all $k = 1, 2, 3$, the random variable α^k is distributed unformly on $[0, 2r]$.

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We have also the following nice property exploiting the simple geometry of the triangle.

Step 7. For all M in S, $\alpha^1\beta^1 + \alpha^2\beta^2 + \alpha^3\beta^3 = r$.

We are now in position to define a solution to the mathematical problem defined above and therefore an optimal strategy. Define the random vector $(\alpha^1, \alpha^2, \alpha^3)$ as above Then, for $k = 1, 2, 3$, let

$$a^j = \frac{M\theta^j}{n^j r}\alpha^k$$

for each voter in group $j \in S^k$. From step 7, $\sum_{j=1}^{J} n^j a^j = M$ and from step 6 the distribution of a^j is uniform on $\left[0, \frac{2\theta^j}{n^j}M\right]$. The probability law p of the random vector a is therefore an optimal strategy. Hereafter, we call it the *disk strategy associated with the partition* (S^1, S^2, S^3). For the situation (ii) in step 5, a similar construction can be done in replacing the triangle in step 6 by a unit square.

There are as many disk solutions as there are ways to choose a partition (S^1, S^2, S^3) satisfying (i) in step 5. We should note that the random variables a^j are far from being independent : not only they sum up to M but from the construction of the disk solution, it is clear that the knowledge of two of them is enough to deduce the value of the $J-2$ remaining ones. *The support of the solution is a two dimensional manifold !* This does not preclude the existence of other solutions. In the symmetric case where $\theta^j = n^j = \frac{1}{J}$ for all $j = 1, ..., J$, Borel (1938) found another solution whose support is the all set $Hex(\theta)$; a generalization of the "hexagonal" solution was discovered by Laslier and Picard (1999) for the case where $J = 4$; the support of the solution is $Hex(\theta)$, in that case an icosahedron. This shows that there are solutions with a support having more than two dimensions. In the symmetric case, other two dimensional solutions exploiting the properties of regular J-gons and avoiding the partition into three subsets, can be constructed, as shown in Laslier and Picard.

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A complete analysis of the inequality induced by the disk solution(s) is rather tricky in our asymmetric framework. For the time being, note that *the expected amount received by a citizen is proportional to the fraction of swing voters in his group and inversely proportional to the relative size of his group.* The fact that groups with more swing voters are favored in expected terms is a confirmation of the analysis of probabilistic voting for the cases where the analysis in pure strategies does not apply. Note that when $\theta^j = n^j$ for all $j = 1, ..., J$, which is the setting considered by Laslier (2002), a^j is uniform on $[0, 2M]$ i.e. large groups are not favored at the detriment of smaller groups. This shows that the tyranny of the majority may be quite limited.The computation of the Lorenz curve induced by a for an arbitrary vector θ is not easy and we will comment that during the presentation.

The equilibrium analysis is complicated by the fact that multiple equilibria exist and very little is known about their common features. For instance, we do not know whether the property on marginals described in step 4 is necessary.

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