PCA for Point Processes

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Plan

Functional Principal Component Analysis (FPCA)

- Principal Component Analysis or PCA is a dimensionality reduction technique for data sets with many features or dimensions. In particular PCA is extensively used for data visualization, by providing a "picture" of high-dimensional data in two or three dimensions, making it easier to interpret.
- FPCA deals with functions. See Ramsay and Silverman (2005) and Hsing and Eubank (2015)
- We consider a $\mathbb{L}_2$-centered random function Z on $[0;1]$. We wish to obtain the "best" representation of Z on an orthonormal basis $\psi = (\psi_d)_{d\in\mathbb{N}^*}$ of $\mathbb{L}_2[0,1]$:

$$Z = \sum_{d=1}^{+\infty} \langle Z, \psi_d \rangle \psi_d$$

(considered bases are not random).
It means that for any other orthonormal basis $\phi = (\phi_d)_{d\in\mathbb{N}^*}$ of $\mathbb{L}_2[0,1]$

$$\mathbb{E}\left[\left\| Z - \sum_{d=1}^{D} \langle Z, \psi_d \rangle \psi_d \right\|^2\right] \le \mathbb{E}\left[\left\| Z - \sum_{d=1}^{D} \langle Z, \phi_d \rangle \phi_d \right\|^2\right], \quad \forall D \in \mathbb{N}^*.$$

FPCA: Mercer theorem

To solve the minimization problem, we consider the covariance kernel associated with Z:

$$K(s,t) = \mathbb{E}[Z(s)Z(t)], \quad 0 \le s,t \le 1 \qquad \text{(remember } \mathbb{E}[Z]=0)$$

and the covariance operator:

$$\begin{aligned} \Gamma_K : \quad \mathbb{L}_2[0,1] &\longmapsto \mathbb{L}_2[0,1] \\ f &\longmapsto \int_0^1 K(s,\cdot) f(s) ds, \end{aligned}$$

The operator Γ_K is self-adjoint, positive-definite and compact. We can apply the spectral theorem, from which we deduce:

Theorem (Mercer representation)

Assume that K is a continuous kernel on $[0,1]\times[0,1]$. Then there exists an orthonormal basis $\psi = (\psi_d)_{d\in\mathbb{N}^*}$ consisting of eigenfunctions of Γ_K such that the corresponding sequence of eigenvalues $\lambda = (\lambda_d)_{d\in\mathbb{N}^*}$ is nonnegative. The eigenfunctions corresponding to non-zero eigenvalues are continuous on $[0,1]$ and K has the representation:

$$K(s,t) = \sum_{d=1}^{\infty} \lambda_d \psi_d(s)\,\psi_d(t), \quad 0 \le s,t \le 1.$$

where the convergence is absolute and uniform.

FPCA: Karhunen-Loève representation

- Under assumptions of the Mercer theorem,

$$\lim_{D\to+\infty}\sup_{t\in[0,1]}\mathbb{E}\left[\left|Z(t)-\sum_{d=1}^{D}\sqrt{\lambda_d}\xi_d\psi_d(t)\right|^2\right]=0,\quad \xi_d=\frac{\langle Z,\psi_d\rangle}{\sqrt{\lambda_d}}$$

  We obtain the Karhunen-Loève representation:

$$Z(t)=\sum_{d=1}^{+\infty}\sqrt{\lambda_d}\xi_d\psi_d(t),\quad t\in[0,1]$$

  The sequence $\xi=(\xi_d)_{d\in\mathbb{N}^*}$ is a sequence of non-correlated centered random variables of variance 1, called the scores.

- Assume w.l.o.g. $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_d\geq\cdots$. For any orthonormal basis $(\phi_d)_{d\in\mathbb{N}^*}$

$$\mathbb{E}\left[\left\|Z-\sum_{d=1}^{D}\langle Z,\psi_d\rangle\psi_d\right\|^2\right]\leq\mathbb{E}\left[\left\|Z-\sum_{d=1}^{D}\langle Z,\phi_d\rangle\phi_d\right\|^2\right],\quad\forall D\in\mathbb{N}^*.$$

- The Karhunen-Loève representation of Z is then the central tool for representation and visualization of the process Z

FPCA: Summary

For the analysis of a stochastic process Z on $[0,1]$ such that

$$\mathbb{E}\left[\|Z\|^2\right] < \infty, \quad \mathbb{E}[Z] = 0.$$

we consider its kernel K assumed to be continuous

$$K(s,t) = \mathbb{E}[Z(s)Z(t)]$$

Mercer expansion:

$$K(s,t) = \sum_{d=1}^{\infty} \lambda_d \psi_d(s)\, \psi_d(t)$$

Karhunen-Loève representation:

$$Z(t) = \sum_{d=1}^{+\infty} \sqrt{\lambda_d} \xi_d \psi_d(t)$$

where in these decompositions $(\lambda_d, \psi_d)_{d\in\mathbb{N}^*}$ are eigenelements of the operator

$$\Gamma(f)(t) = \int_0^1 K(s,t) f(s) ds, \quad f \in \mathbb{L}_2[0,1]$$

Illustration: the Brownian motion on $[0, 1]$

For the analysis of the Brownian motion, W, on $[0, 1]$, we easily obtain for $0 \leq s, t \leq 1$,

$$K(s, t) = \min(s, t) = \sum_{d=1}^{\infty} \lambda_d \psi_d(s) \, \psi_d(t),$$

with

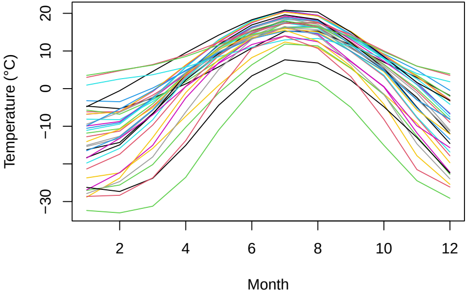
$$\lambda_d = \frac{1}{\pi^2 (d - \frac{1}{2})^2}, \quad \psi_d(t) = \sqrt{2} \sin\left((d - \frac{1}{2})\pi t\right), \qquad d \geq 1$$

and

$$W(t) = \sqrt{2} \sum_{d=1}^{\infty} \xi_d \frac{\sin\left(\left(d - \frac{1}{2}\right) \pi t\right)}{\left(d - \frac{1}{2}\right) \pi}.$$

The sequence $\xi = (\xi_d)_{d \in \mathbb{N}^*}$ is a sequence of non-correlated centered random variables of variance 1 with Gaussian distribution.

Statistical illustration: analysis of temperature curves. (I)



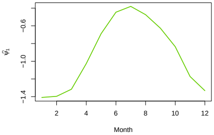
Source: Ramsay and Silverman (2005) and Ramsay, Hooker and Graves (2009)

- Each curve is the evolution of the mean temperature of a Canadian city. We denote by Z_i the curve corresponding to the i-th city and we model $Z_1, \dots, Z_n \sim_{i.i.d.} Z$
- Karhunen-Loève representation of Z :

$$Z(t) = \mathbb{E}[Z(t)] + \sum_{d \geq 1} \sqrt{\lambda_d} \xi_j \psi_d(t).$$

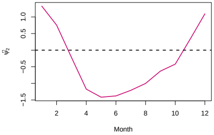
Statistical illustration: analysis of temperature curves (II)

$$Z(t) = \mathbb{E}[Z(t)] + \sum_{d\geq 1} \sqrt{\lambda_d}\xi_d\psi_d(t)$$
$$= \mathbb{E}[Z(t)] + \sqrt{\lambda_1}\xi_1{\color{green}\psi_1}(t) + \sqrt{\lambda_2}\xi_2{\color{red}\psi_2}(t) + \sum_{d\geq 3} \sqrt{\lambda_d}\xi_d\psi_d(t)$$



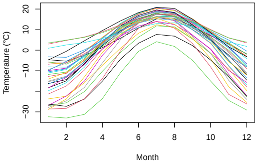
For each city i,

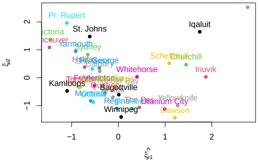
$${\color{blue}Z_i(t)} = \mathbb{E}[Z(t)] + \sqrt{\lambda_1}{\color{blue}\xi_{i1}}{\color{green}\psi_1}(t) + \sqrt{\lambda_2}{\color{blue}\xi_{i2}}{\color{red}\psi_2}(t) + \sum_{d\geq 3} \sqrt{\lambda_d}{\color{blue}\xi_{id}}\psi_d(t)$$

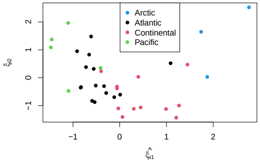


Every term of the expansion has to be estimated.

We estimate $\xi_{id} = \langle Z_i - \mathbb{E}[Z(t)], \psi_d \rangle / \sqrt{\lambda_d}$, by $\widehat{\xi}_{id} = \langle Z_i - \overline{Z}_n, \widehat{\psi}_d \rangle / \sqrt{\widehat{\lambda}_d}$.



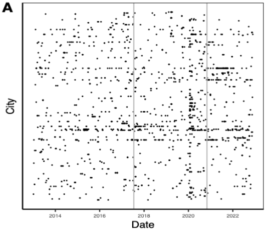




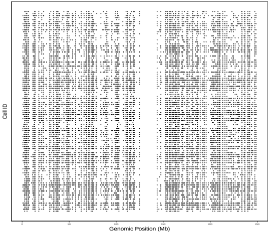
Plan

Our contribution: PCA for Point Processes

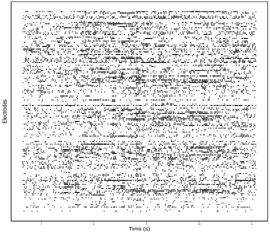
Earthquakes data



Single-Cell Chromatin data



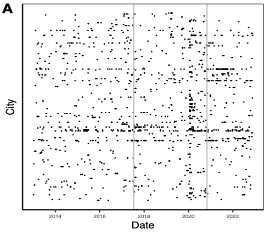
Neuronal spike data



- A (temporal) point process $N = (N_t)_t$ is a random countable set of points of $\mathbb{R}$
- For dimension reduction and vizualization purposes, can we develop a specific framework for point processes?
- In the sequel, we assume that we observe n i.i.d. point processes $(N^1, N^2, \ldots, N^n)$ with the same distribution as N.

PCA for point processes: state of the art

Earthquakes data



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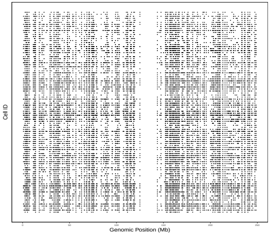

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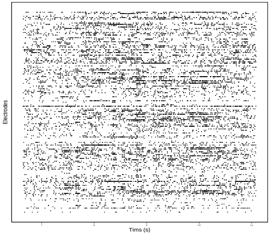

- Illian, Benson, Crawford and Staines (2006), Manté, Yao and Degiovanni (2007) and Wu, Müller and Zhang (2013) proposed to apply FPCA to the intensity processes associated with point processes. Note that these intensities are not observed.
- Carrizo Vergara (2022) proposed a framework to get series expansions for general random measures.

Karhunen-Loève and Mercer theorems for PP (I)

- Considering n i.i.d. temporal point processes $(N_1, N_2, \dots, N_n)$, observed on the time interval $[0,1]$, we set for any Borelian set B,

$$\Pi_i(B) = \sum_{T \in N_i} \mathbf{1}_{\{T \in B\}} = \mathrm{Card}(N_i \cap B) \ (\in \mathbb{N}).$$

  Hence, $\Pi_1, \dots, \Pi_n$ is a sample of random measures with the same distribution as Π.
- We assume that $\mathbb{E}[\Pi^2([0,1])] < +\infty$ and we set for any B and B',

$$\Delta(B) = \Pi(B) - m(B), \quad m(B) = \mathbb{E}\big[\Pi(B)\big]$$

  and

$$C_\Delta(B \times B') = \mathrm{Cov}(\Pi(B), \Pi(B')) = \mathbb{E}\left[\Delta(B)\Delta(B')\right]$$

- To use standard tools of fPCA and we introduce for any random measure μ

$$F_\mu(t) = \mu([0,t]), \quad t \in [0,1].$$

  and

$$K_\Delta(s,t) = C_\Delta([0,s] \times [0,t]) = \mathbb{E}[F_\Delta(s) F_\Delta(t)], \quad s,t \in [0,1]$$

- Assuming that K_Δ is continuous, Mercer's theorem applies and K_Δ writes:

$$K_\Delta(s,t) = \sum_{j \geq 1} \lambda_j \psi_j(s) \psi_j(t), \quad s,t \in [0,1]$$

Karhunen-Loève and Mercer theorems for PP (II)

$$K_\Delta(s,t) = \sum_{j\geq 1} \lambda_j \psi_j(s)\psi_j(t), \quad s,t \in [0,1]$$

Theorem

We assume that $\mathbb{E}[\Pi^2([0,1])] < +\infty$ *and* K_Δ *is continuous. Then*

1. *There exists a signed measure* μ_j *that verifies*

$$\psi_j(t) = \mu_j([0,t]) = F_{\mu_j}(t), \quad t \in [0,1]$$

2. *Karhunen-Loève expansion: there exists a sequence* $\{\xi_j\}_{j\geq 1}$ *of uncorrelated real random variables of mean zero and variance one such that*

$$\lim_{J\to+\infty} \mathbb{E}\left[\left\|\Pi - m - \sum_{j=1}^{J} \sqrt{\lambda_j}\xi_j\mu_j\right\|^2_{\mathcal{H}^{-1}}\right] = 0$$

3. *Mercer theorem for* C_Δ*:*

$$\lim_{J\to+\infty} \left\|C_\Delta - \sum_{j=1}^{J} \lambda_j \mu_j \otimes \mu_j\right\|_{\mathcal{H}^{-2}} = 0$$

Karhunen-Loève and Mercer theorems for PP (III)

- Convergence in a strong sense: on $\mathcal{H}^{-1}$ for KL and $\mathcal{H}^{-2}$ for Mercer with

$$\|\mu\|_{\mathcal{H}^{-k}} = \sup\left\{|\langle f,\mu\rangle| : f \in \mathcal{H}_0^k \text{ and } \sum_{|\alpha|\le k} \|\partial_\alpha f\|^2 \le 1\right\}.$$

and

$$\mathcal{H}_0^k = \left\{f \in \mathbb{L}^2(I) : \partial_\alpha f \in \mathbb{L}^2(I) \text{ for all } |\alpha| \le k\right\}$$

- Consequence: For Π_i the random measure associated with N_i,

$$\Pi_i(B) = m(B) + \sum_{j=1}^{+\infty} \sqrt{\lambda_j}\xi_{i,j}\mu_j(B), \qquad B \in \mathcal{B}$$

or

$$\Pi_i([0,t]) = m([0,t]) + \sum_{j=1}^{+\infty} \sqrt{\lambda_j}\xi_{i,j}F_{\mu_j}(t), \qquad t \ge 0$$

Terminology:
- The μ_j's are called the principal measures
- The $\xi_{i,j}$ are the scores associated with the Π_i's. We easily show:

$$\xi_{i,j} = \frac{\langle F_{\mu_j}, F_{\Delta_i}\rangle}{\sqrt{\lambda_j}}.$$

Principal elements for the Poisson Process

We consider $(N_1, N_2, \ldots, N_n)$ n i.i.d. Poisson processes with intensity function w assumed to be continuous and positive on $(0,1)$.

$$\mathrm{Card}(N_i \cap [0,t]) = \Pi_i([0,t]) = \int_0^t w(u)du + \sum_{j=1}^{+\infty} \sqrt{\lambda_j}\xi_{i,j} F_{\mu_j}(t), \qquad t \in [0,1],$$

We can derive the kernel expression: $K_\Delta(s,t) = \int_0^{\min\{s;t\}} w(u)du,\ 0 \leq s,t \leq 1.$

Theorem

We take $j \geq 1$.

1. *Homogeneous Poisson process: If w is constant: $w \equiv w_0$*

$$\lambda_j = \frac{w_0}{\pi^2 (j - \frac{1}{2})^2}, \quad F_{\mu_j}(t) = \sqrt{2}\sin(\pi(j-1/2)t), \quad t \in [0,1]$$

2. *Inhomogeneous Poisson process: If w is not constant, we have for $j \geq 1$:*

$$\lambda_j \overset{j \to +\infty}{\sim} \frac{\left(\int_0^1 \sqrt{w(u)}du\right)^2}{\pi^2 j^2}, \quad F_{\mu_j} \text{ has at least } j \text{ zeros on } [0,1).$$

Principal elements for Hawkes processes (I)

We consider $(N_1, N_2, \ldots, N_n)$ n i.i.d. Hawkes processes with intensity function

$$w(t) = w_0 + \alpha \int_{-\infty}^{t^-} \exp(-\beta(t-s))dN_s = w_0 + \alpha \sum_{T \in N,\ T<t} \exp(-\beta(t-T)),$$

with $w_0 > 0$ and $0 < \alpha < \beta$. Let $w_1 = \frac{\beta w_0}{\beta - \alpha} = \frac{w_0}{1-\alpha/\beta}$.

$$\mathrm{Card}(N_i \cap [0,t]) = \Pi_i([0,t]) = w_1 t + \sum_{j=1}^{+\infty} \sqrt{\lambda_j} \xi_{i,j} F_{\mu_j}(t), \qquad t \in [0,1],$$

Theorem

Assume $\alpha < \beta$ and

$$\frac{\alpha(2\beta-\alpha)}{\beta-\alpha}\left(2 + \frac{3 - 2e^{-(\beta-\alpha)/2} - e^{-(\beta-\alpha)}}{\beta-\alpha}\right) < 1.$$

There exist C_1 and C_2 two constants not depending on j such that fro any $j \geq 1$,

$$\left|\lambda_j - \frac{w_1}{\pi^2 (j-1/2)^2}\right| \leq C_1 j^{-4}, \qquad \sup_{t\in[0,1]} \left|F_{\mu_j}(t) - \sqrt{2}\sin(\pi(j-1/2)t)\right| \leq C_2 j^{-1}$$

Principal elements for Hawkes processes (I)

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Theorem

Assume $\alpha < \beta$ and

$$\underbrace{\frac{\alpha(2\beta-\alpha)}{\beta-\alpha}}_{=\beta \times \frac{r(2-r)}{1-r},\ r=\alpha/\beta} \underbrace{\left(2 + \frac{3 - 2e^{-(\beta-\alpha)/2} - e^{-(\beta-\alpha)}}{\beta-\alpha}\right)}_{\in[2,4]} < 1$$

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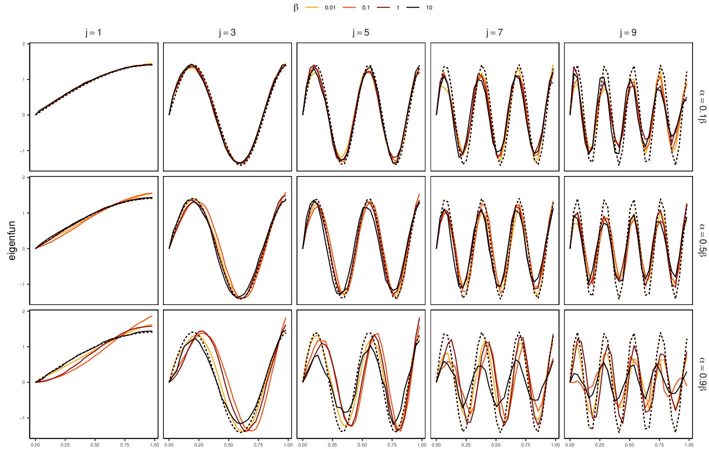
Principal elements for Hawkes processes (II)



Eigenfunctions for Hawkes Processes with different transfert functions.
Dotted lines: asymptotic eigenfunctions $t \longmapsto \sqrt{2}\sin(\pi(j-1/2)t))$.
First row: $\alpha = 0.1\beta$; Second row: $\alpha = 0.5\beta$; Third row: $\alpha = 0.9\beta$

Principal elements for Hawkes processes (III)

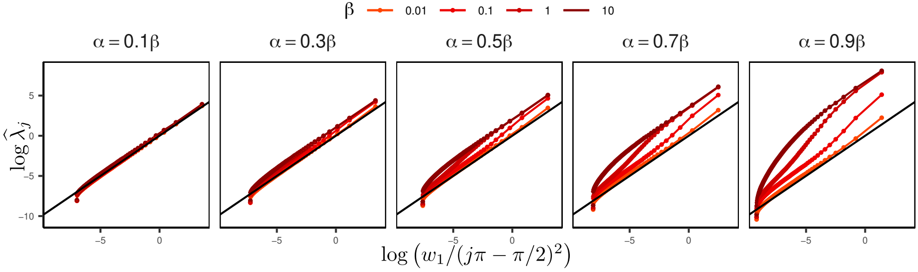


Figure: Eigenvalues (log-scale) for Hawkes Processes over 50 replicated. Each dot corresponds to a value of $j \in \{1, \ldots, 50\}$. The empirical average is plotted vs the expected theoretical asymptotic regime of eigenvalues in $w_1/(j\pi - \pi/2)^2$, as expected. The black line corresponds to the first bisector, so that the points align if the empirical convergence matches the theoretical regime.

Each column corresponds to a value of $r = \frac{\beta}{\alpha}$: $r = 0.1$ $r = 0.3$ $r = 0.5$ $r = 0.7$ $r = 0.9$

Sketch of the proofs

To determine the (λ_j, F_{μ_j})'s, remember that

$$K_\Delta(s,t) = \sum_{j\geq 1} \lambda_j \psi_j(s)\psi_j(t), \quad s,t \in [0,1]$$

and $F_{\mu_j} = \psi_j$. So, we use the covariance operator Γ_Δ associated with K_Δ:

$$\Gamma_\Delta(f)(\cdot) = \int_0^1 K_\Delta(\cdot,t)f(t)dt, \qquad f \in \mathbb{L}^2,$$

whose eigenelements are the (λ_j, F_{μ_j})'s. We finally show:

Theorem

We set $H_j(t) = \int_t^1 F_{\mu_j}(s)ds$, $t \in [0,1]$. Then, $(\lambda_j, F_{\mu_j})_{j\geq 1}$ are the eigenelements of the operator Γ_Δ if and only if $(\lambda_j, H_j)_{j\geq 1}$ are solutions of the following system:

$$\begin{cases} -\lambda y''(t) = w(t)y(t) + \int_0^1 M(s,t)y(s)ds, \quad t \in (0,1), \\ y(1) = 0, \quad y'(0) = 0. \end{cases}$$

- For the Poisson process, w is the intensity of the process and $M = 0$.
- For the Hawkes process with exponential self-exciting function, $w(t) = w_1$ and M is an exponential convolution kernel.

Estimation of eigenelements

Recall that the $(\lambda_j, \psi_j)_{j\geq 1}$'s are the eigenelements of Γ_Δ, with

$$\Gamma_\Delta(f)(s) = \int_0^1 K_\Delta(s,t) f(t) dt, \quad \Delta(B) = \Pi(B) - \mathbb{E}[\Pi(B)]$$

and

$$K_\Delta(s,t) = C_\Delta([0,s]\times[0,t]), \quad C_\Delta(B \times B') = \mathrm{Cov}(\Pi(B), \Pi(B')).$$

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We set

$$\widehat{m} = \frac{1}{n}\sum_{i=1}^n \Pi_i$$

$$\widehat{K}_{\widehat{\Delta}}(s,t) = \widehat{C}_{\widehat{\Delta}}([0,s]\times[0,t]), \quad \widehat{C}_{\widehat{\Delta}}(B\times B') = \frac{1}{n}\sum_{i=1}^n \sum_{T,T'\in N_i} \mathbf{1}_{\{(T,T')\in B\times B'\}} - \widehat{m}(B)\times\widehat{m}(B')$$

and finally

$$\widehat{\Gamma}_{\widehat{\Delta}}(f)(s) = \int_0^1 \widehat{K}_{\widehat{\Delta}}(s,t)f(t)dt.$$

Eigenelements of Γ_Δ are estimated by eigenelements of $\widehat{\Gamma}_{\widehat{\Delta}}$.

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Eigenelements of Γ_Δ are estimated by eigenelements of $\widehat{\Gamma}_{\widehat{\Delta}}$.

- Can we compute eigenelements of $\widehat{\Gamma}_{\widehat{\Delta}}$?
- Do these estimates achieve optimal rates?

Computing eigenelements of $\widehat{\Gamma}_{\widehat{\Delta}}$

- Consider all occurrences sorted in non-decreasing order:

$$\mathcal{T} = \bigcup_{i=1}^{n} N_i \bigcup \{0; 1\} = \Big\{0,\, T_1,\, T_2, \ldots, 1\Big\}$$

- We build the histogram system associated with this grid:

$$e_\ell(t) = \frac{1}{\sqrt{T_\ell - T_{\ell-1}}} 1_{[T_{\ell-1}; T_\ell)}(t), \quad \ell = 1, \ldots, |\mathcal{T}|,$$

- Let

$$\widehat{G}_{\widehat{\Delta}} = \left(\langle \widehat{\Gamma}_{\widehat{\Delta}} e_\ell, e_{\ell'} \rangle\right)_{1 \le \ell, \ell' \le |\mathcal{T}|}.$$

  - The matrix $\widehat{G}_{\widehat{\Delta}}$ is constructed explicitly from the data.
  - The eigenvalues of $\widehat{G}_{\widehat{\Delta}}$ coincide with the ones of $\widehat{\Gamma}_{\widehat{\Delta}}$.
  - The eigenfunction $\widehat{\psi}_j$ of $\widehat{\Gamma}_{\widehat{\Delta}}$ is constructed explicitly from the eigenvector $\widehat{v}_j$ associated to the j-th largest eigenvalue of $\widehat{G}_{\widehat{\Delta}}$:

$$\widehat{\psi}_j = \sum_{\ell=1}^{|\mathcal{T}|} \widehat{v}_{j,\ell} e_\ell.$$

Convergence rates

Theorem

Assume $C_4 = \mathbb{E}\left[\|F_\Delta\|^4\right] < +\infty$ *and the eigenvalues* $(\lambda_j)_j$ *are simple. Then, we have:*

$$\mathbb{E}\left[\sup_{j\geq 1}|\widehat{\lambda}_j - \lambda_j|^2\right] \leq 4\frac{C_4}{n}.$$

For all $j \geq 1$,

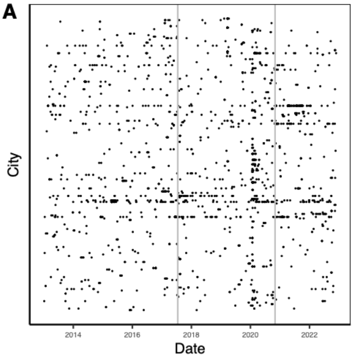
$$\mathbb{E}\left[\|\widehat{\psi}_j - \mathrm{sign}(\langle\widehat{\psi}_j, \psi_j\rangle)\psi_j\|^2\right] \leq 32\delta_j^{-2}\frac{C_4}{n}.$$

where $\delta_j = \min\{\lambda_j - \lambda_{j-1}; \lambda_j - \lambda_{j+1}\}$ *for* $j \geq 2$ *and* $\delta_1 = \lambda_1 - \lambda_2$.

$$\mathbb{E}[\|\widehat{\mu}_j - \mathrm{sign}(\langle\widehat{\psi}_j, \psi_j\rangle\mu_j\|^2_{\mathcal{H}^{-1}((0,1))}] \leq 32\delta_j^{-2}\frac{C_4}{n}.$$

We estimate the scores $\xi_{i,j} = \frac{\langle\psi_j, F_{\Delta_i}\rangle}{\sqrt{\lambda_j}}$ by $\widehat{\xi}_{i,j} = \frac{\langle\widehat{\psi}_j, F_{\widehat{\Delta}_i}\rangle}{\sqrt{\widehat{\lambda}_j}}$ if $\widehat{\lambda}_j > 0$.

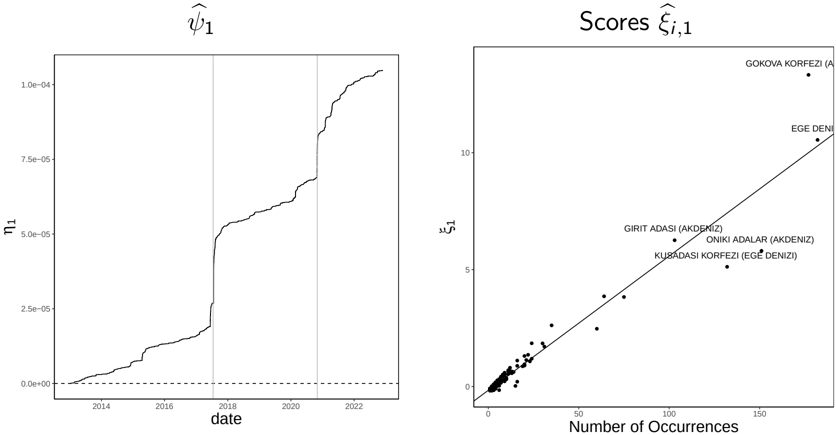
Application to the exploration of earthquakes



Earthquake occurrences in Turkey and neighboring regions of Greece, recorded between January 2013 and January 2023 in 195 cities. Each line corresponds to a city and each dot to an earthquake occurrence.

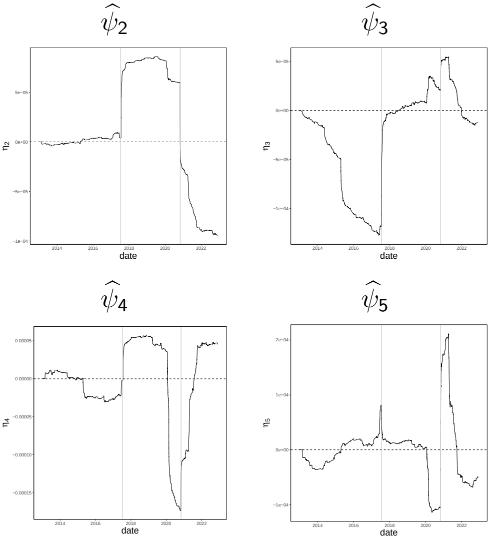
Source : `http://www.koeri.boun.edu.tr/sismo/2/earthquake-catalog/`

Study of axis 1

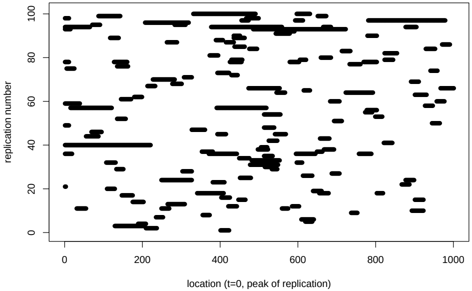


Breakpoint dates (grey vertical lines) correspond to 2017.07.16 and 2020.11.01.
Scores on $\widehat{\mu}_1$ are highly correlated with the total number of events of the process.

Study of axes 2–5



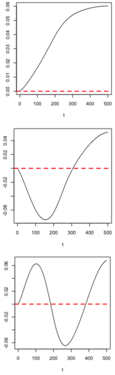
Application to the genomic data



$\hat{\psi}_1$

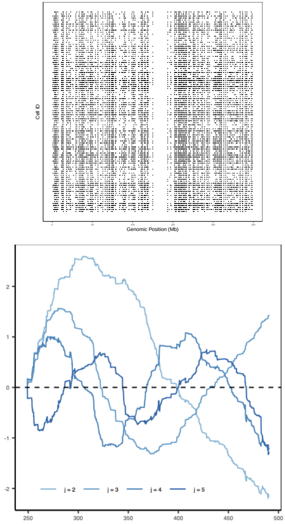
$\hat{\psi}_2$

$\hat{\psi}_3$

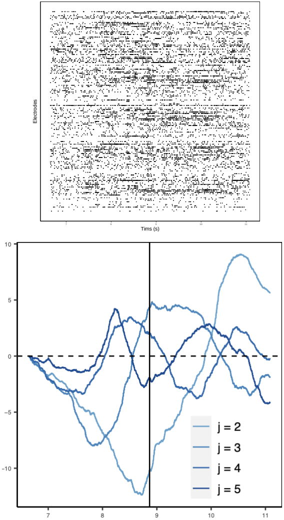


G-quadruplexes are secondary structures in DNA or RNA and serve specific functional purposes. The DNA is split so that each line represents a replication origin and each point an occurence of a G-quadruplex.

Eigenfunctions are oscillating functions



Biological single cell data



Neuronal spike data

Conclusions

- Conclusions:
  - We provide a framework to perform PCA for Point Processes
  - Karhunen-Loève and Mercer theorems for Point Processes are established
  - Principal measures for Poisson and Hawkes processes are studied
  - Parametric convergence rates for the estimators of principal measures are obtained

- Reference: PICARD F., RIVOIRARD V., ROCHE A. AND PANARETOS V. (2024) *PCA for Point Processes*. In revision. arXiv:2404.19661

Thank you for your attention.
Questions and remarks are welcomed!