On the impossibility of detecting a late change-point in the preferential attachment random graph model

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The talk and the current work is inspired from

Gianmarco Bet et al. (2023) **"Detecting a late change-point in the preferential attachment model"**, arXiv :2310.02603

in which the authors study the feasibility of detecting a changepoint in the (affine) Preferential Attachement (PA) model.

Outline

1. Introduction : what is the PA model ? what is a changepoint ?
   Statement of main results and the **conjecture**

2. Feasibility : how to detect a late changepoint.

3. Impossibility : when is it impossible to detect a late change-point ?

1. Introduction : what is the PA model ? what is a changepoint ? Statement of main results and the **conjecture**
2. Feasibility : how to detect a late changepoint.
3. Impossibility : when is it impossible to detect a late change-point ?

Time 0

$s =$	Deg.
0	0

G_0

Time 1

$s =$	Deg.
0	0
1	0

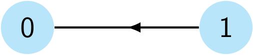
Time 1, Edge #1

$s =$	Deg.	$\Pr(\text{choose } s) \propto$
0	0	1
1	0	

Time 1, Edge #1

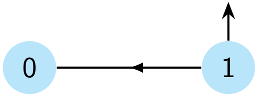
$s =$	Deg.	$\Pr(\text{choose } s) \propto$
0	0	1
1	0	

Time 1, Edge #1



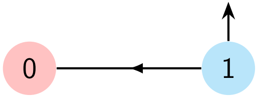
$s =$	Deg.
0	1
1	1

Time 1, Edge #2



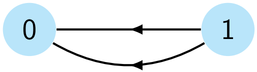
$s =$	Deg.	$\Pr(\text{choose } s) \propto$
0	1	1
1	1	

Time 1, Edge $\#2$



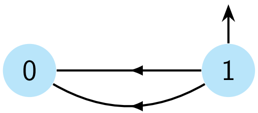
$s =$	Deg.	$\Pr(\text{choose } s) \propto$
0	1	1
1	1	

Time 1, Edge #2



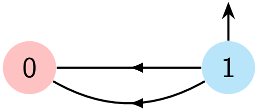
$s =$	Deg.
0	2
1	2

Time 1, Edge #3



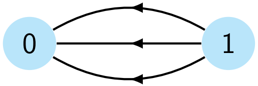
$s =$	Deg.	$\Pr(\text{choose } s) \propto$
0	2	1
1	2	

Time 1, Edge #3



$s =$	Deg.	$\Pr(\text{choose } s) \propto$
0	2	1
1	2	

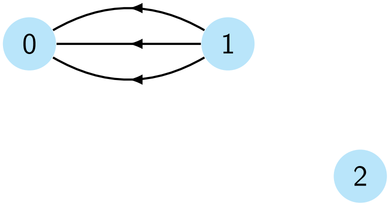
Time 1, Edge #3



$s =$	Deg.
0	3
1	3

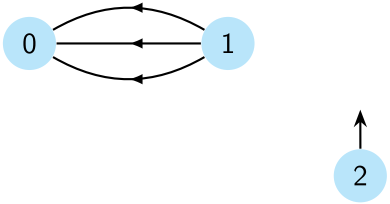
G_1

Time 2



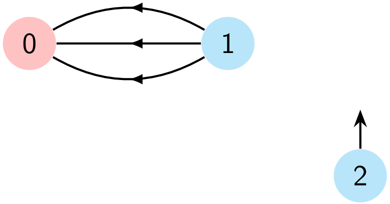
$s =$	Deg.
0	3
1	3
2	0

Time 2, Edge #4



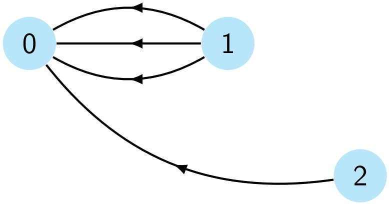
$s =$	Deg.	$\Pr(\text{choose } s) \propto$
0	3	$3 + \delta(2)$
1	3	$3 + \delta(2)$
2	0	

Time 2, Edge #4



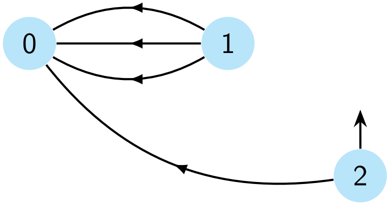
$s =$	Deg.	$\Pr(\text{choose } s) \propto$
0	3	$3+\delta(2)$
1	3	$3+\delta(2)$
2	0	

Time 2, Edge #4



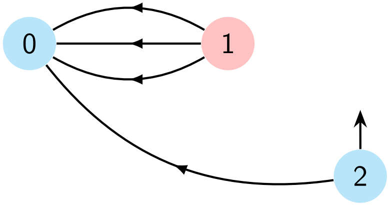
$s =$	Deg.
0	4
1	3
2	1

Time 2, Edge #5



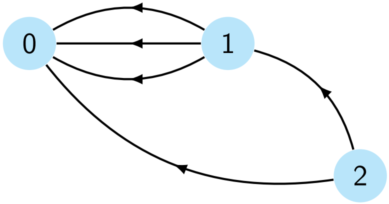
$s =$	Deg.	$\Pr(\text{choose } s) \propto$
0	4	$4 + \delta(2)$
1	3	$3 + \delta(2)$
2	1	

Time 2, Edge #5



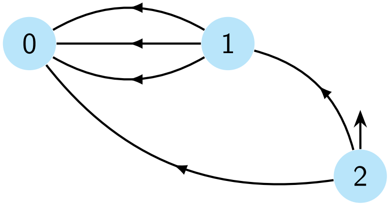
$s =$	Deg.	Pr(choose s) $\propto$
0	4	$4 + \delta(2)$
1	3	$3 + \delta(2)$
2	1	

Time 2, Edge #5



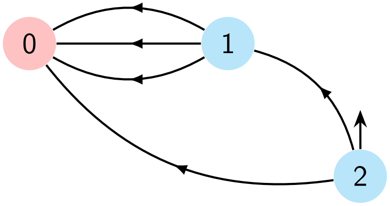
$s =$	Deg.
0	4
1	4
2	2

Time 2, Edge #6



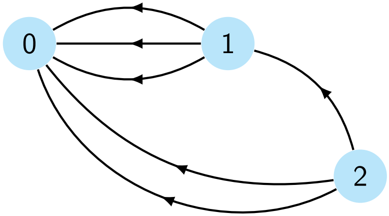
$s =$	Deg.	$\Pr(\text{choose } s) \propto$
0	4	$4 + \delta(2)$
1	4	$4 + \delta(2)$
2	2	

Time 2, Edge #6



$s =$	Deg.	$\Pr(\text{choose } s) \propto$
0	4	$4 + \delta(2)$
1	4	$4 + \delta(2)$
2	2	

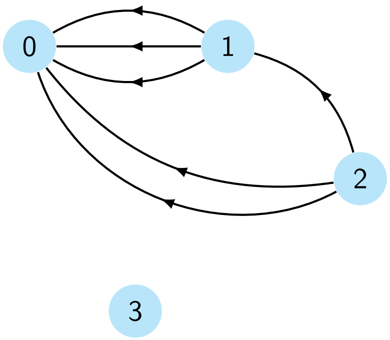
Time 2, Edge #6



$s =$	Deg.
0	5
1	4
2	3

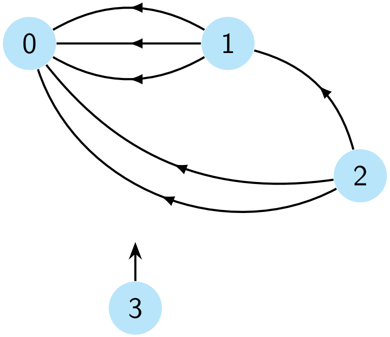
G_2

Time 3



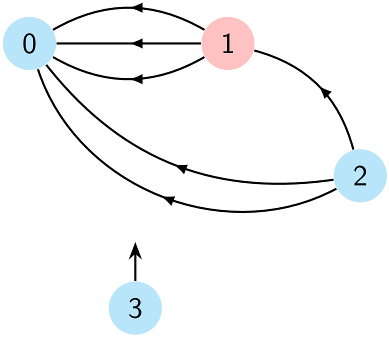
$s =$	Deg.
0	5
1	4
2	3
3	0

Time 3, Edge #7



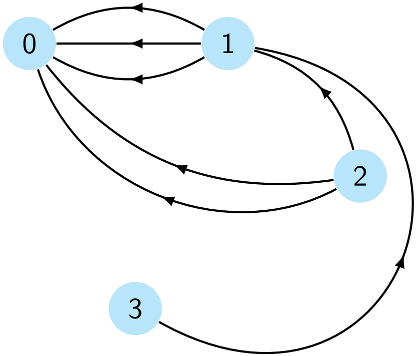
$s =$	Deg.	Pr(choose s) $\propto$
0	5	$5 + \delta(3)$
1	4	$4 + \delta(3)$
2	3	$3 + \delta(3)$
3	0	

Time 3, Edge #7



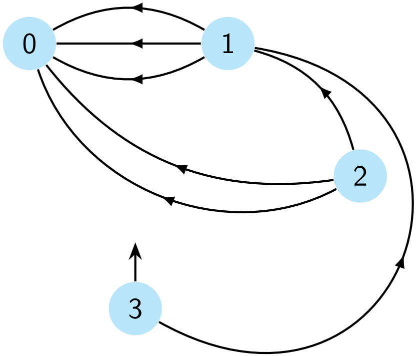
$s =$	Deg.	$\Pr(\text{choose } s) \propto$
0	5	$5 + \delta(3)$
1	4	$4 + \delta(3)$
2	3	$3 + \delta(3)$
3	0	

Time 3, Edge #7



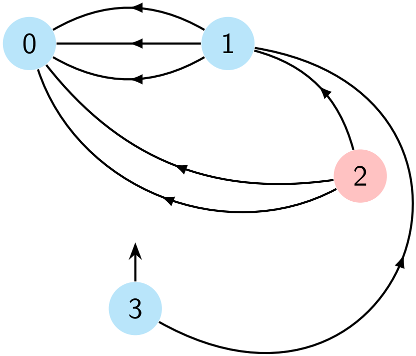
$s =$	Deg.
0	5
1	5
2	3
3	1

Time 3, Edge #8



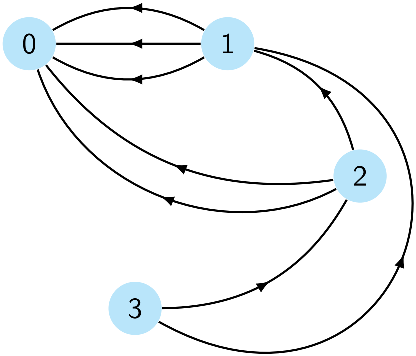
$s =$	Deg.	Pr(choose s) $\propto$
0	5	$5 + \delta(3)$
1	5	$5 + \delta(3)$
2	3	$3 + \delta(3)$
3	1	

Time 3, Edge #8



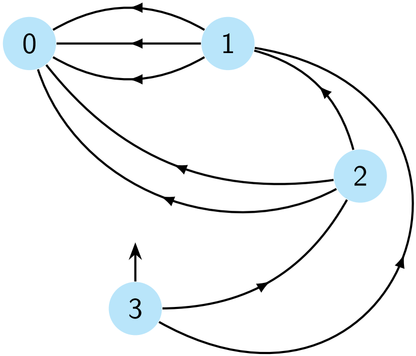
$s =$	Deg.	$\Pr(\text{choose } s) \propto$
0	5	$5 + \delta(3)$
1	5	$5 + \delta(3)$
2	3	$3 + \delta(3)$
3	1	

Time 3, Edge #8



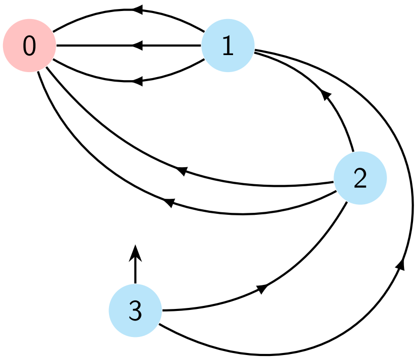
$s =$	Deg.
0	5
1	5
2	4
3	2

Time 3, Edge #9



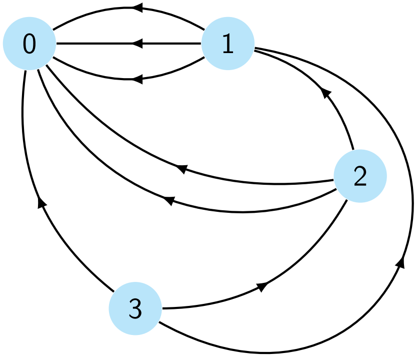
$s =$	Deg.	$\Pr(\text{choose } s) \propto$
0	5	$5 + \delta(3)$
1	5	$5 + \delta(3)$
2	4	$4 + \delta(3)$
3	2	

Time 3, Edge #9



$s =$	Deg.	$\Pr(\text{choose } s) \propto$
0	5	$5 + \delta(3)$
1	5	$5 + \delta(3)$
2	4	$4 + \delta(3)$
3	2	

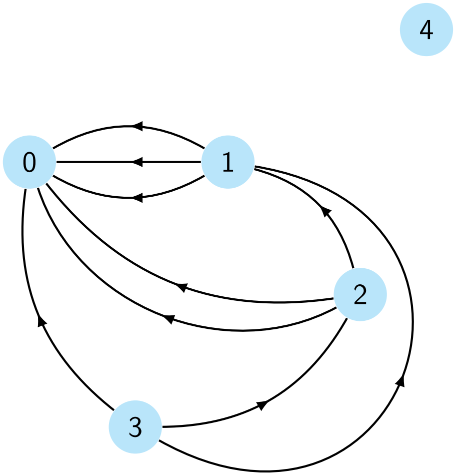
Time 3, Edge #9



$s =$	Deg.
0	6
1	5
2	4
3	3

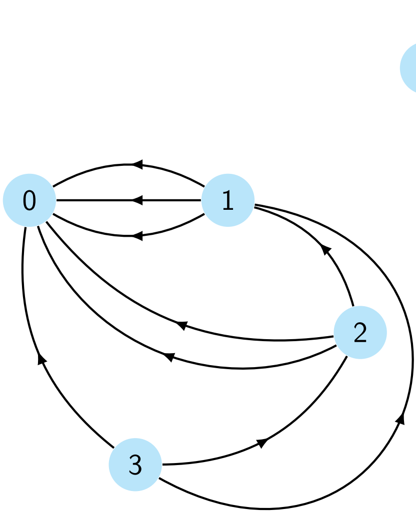
G_3

Time 4



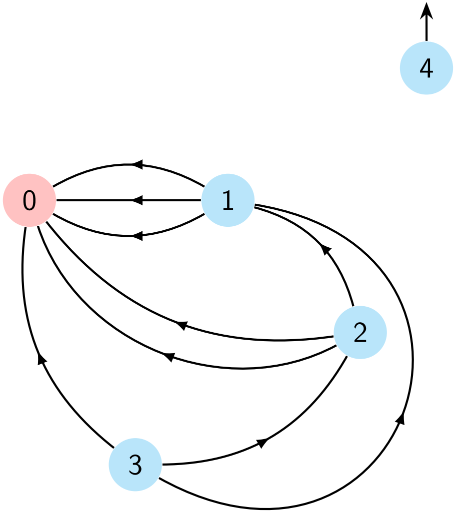
$s =$	Deg.
0	6
1	5
2	4
3	3
4	0

Time 4, Edge #10



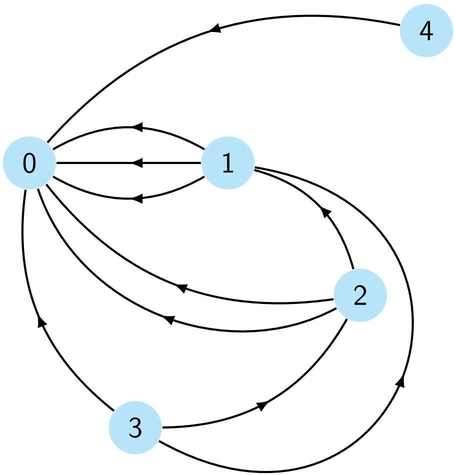
$s =$	Deg.	$\Pr(\text{choose } s) \propto$
0	6	$6 + \delta(4)$
1	5	$5 + \delta(4)$
2	4	$4 + \delta(4)$
3	3	$3 + \delta(4)$
4	0	

Time 4, Edge #10



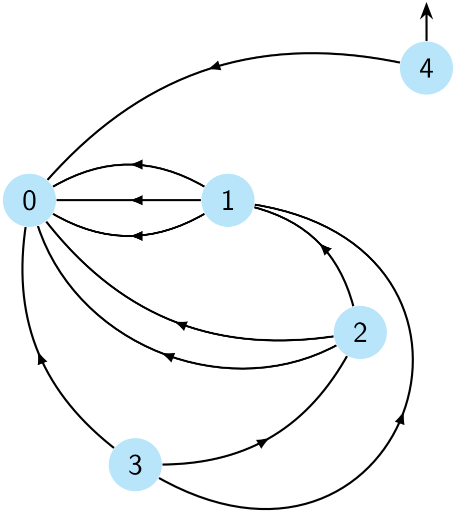
$s =$	Deg.	Pr(choose s) $\propto$
0	6	$6 + \delta(4)$
1	5	$5 + \delta(4)$
2	4	$4 + \delta(4)$
3	3	$3 + \delta(4)$
4	0	

Time 4, Edge #10



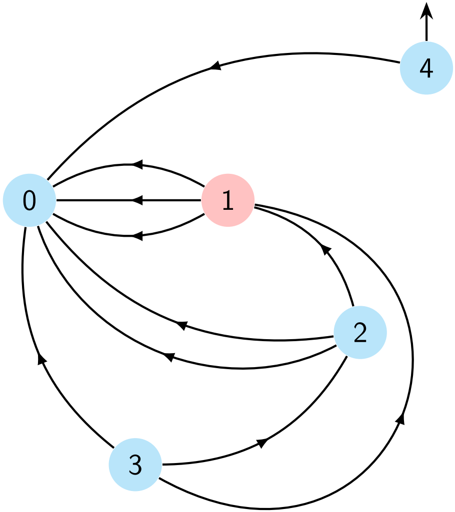
$s =$	Deg.
0	7
1	5
2	4
3	3
4	1

Time 4, Edge #11



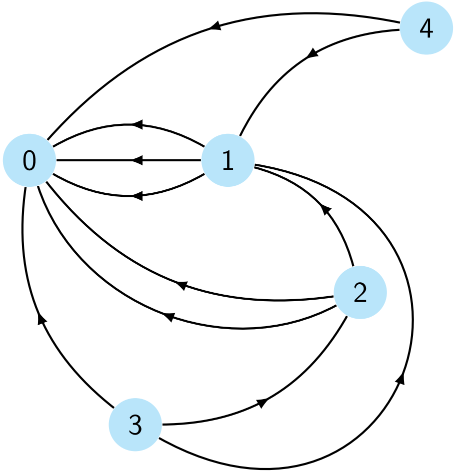
$s =$	Deg.	Pr(choose s) $\propto$
0	7	$7 + \delta(4)$
1	5	$5 + \delta(4)$
2	4	$4 + \delta(4)$
3	3	$3 + \delta(4)$
4	1	

Time 4, Edge #11



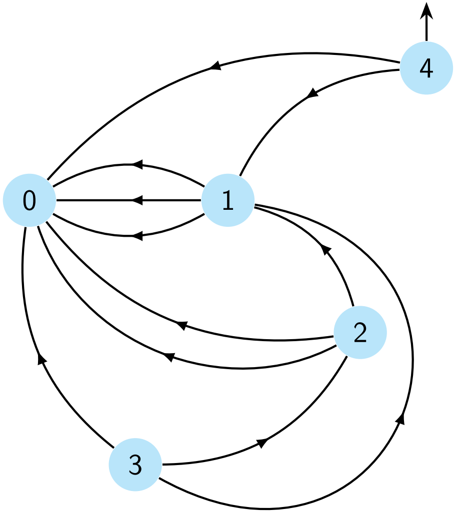
$s =$	Deg.	Pr(choose s) $\propto$
0	7	$7 + \delta(4)$
1	5	$5 + \delta(4)$
2	4	$4 + \delta(4)$
3	3	$3 + \delta(4)$
4	1	

Time 4, Edge #11



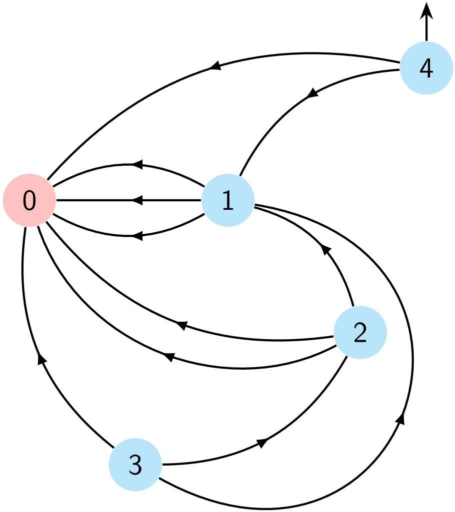
$s =$	Deg.
0	7
1	6
2	4
3	3
4	2

Time 4, Edge #12



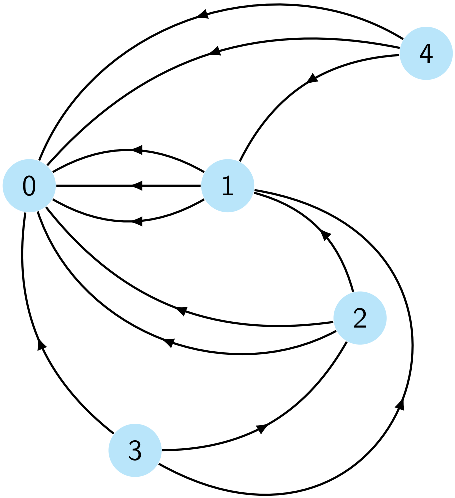
$s =$	Deg.	Pr(choose s) $\propto$
0	7	$7+\delta(4)$
1	6	$6+\delta(4)$
2	4	$4+\delta(4)$
3	3	$3+\delta(4)$
4	2	

Time 4, Edge $\#12$



$s =$	Deg.	Pr(choose s) $\propto$
0	7	$7 + \delta(4)$
1	6	$6 + \delta(4)$
2	4	$4 + \delta(4)$
3	3	$3 + \delta(4)$
4	2	

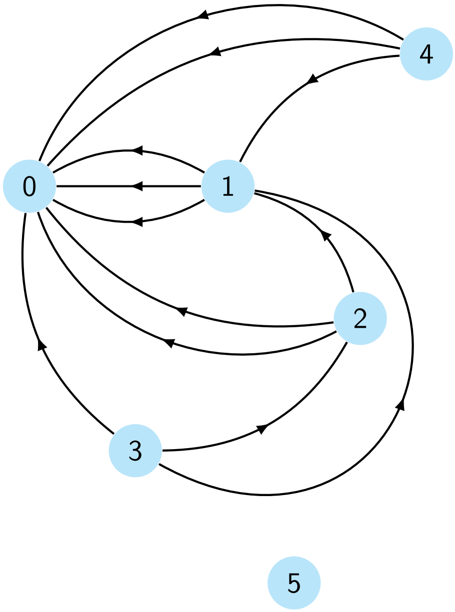
Time 4, Edge #12



$s =$	Deg.
0	8
1	6
2	4
3	3
4	3

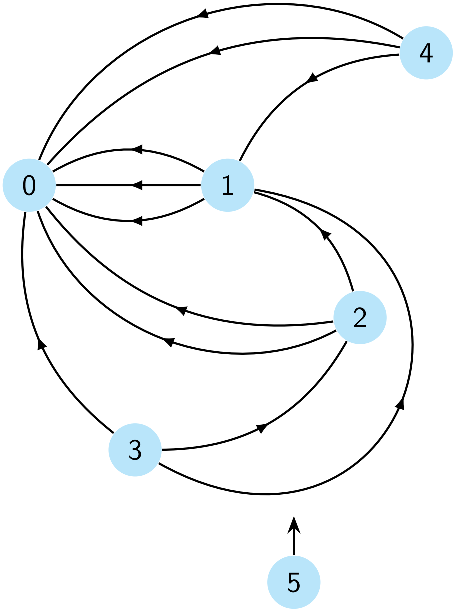
G_4

Time 5



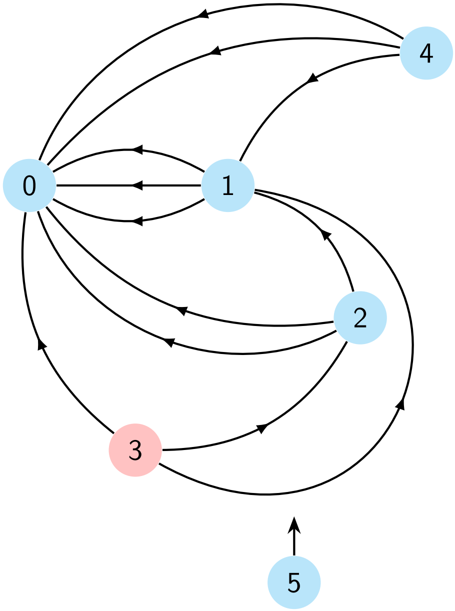
$s =$	Deg.
0	8
1	6
2	4
3	3
4	3
5	0

Time 5, Edge #13



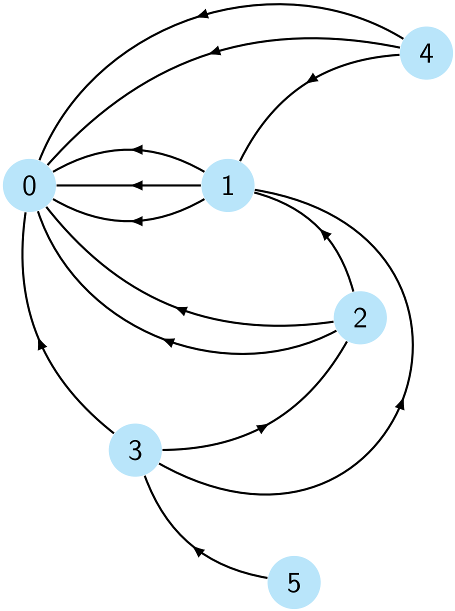
$s =$	Deg.	Pr(choose s) $\propto$
0	8	$8 + \delta(5)$
1	6	$6 + \delta(5)$
2	4	$4 + \delta(5)$
3	3	$3 + \delta(5)$
4	3	$3 + \delta(5)$
5	0	

Time 5, Edge #13



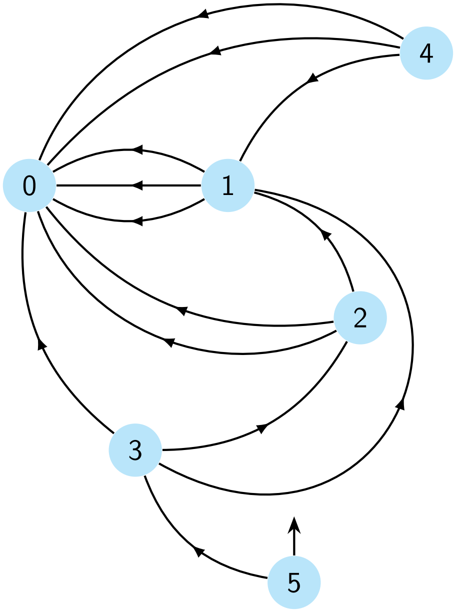
$s =$	Deg.	$\Pr(\text{choose } s) \propto$
0	8	$8 + \delta(5)$
1	6	$6 + \delta(5)$
2	4	$4 + \delta(5)$
3	3	$3 + \delta(5)$
4	3	$3 + \delta(5)$
5	0	

Time 5, Edge #13



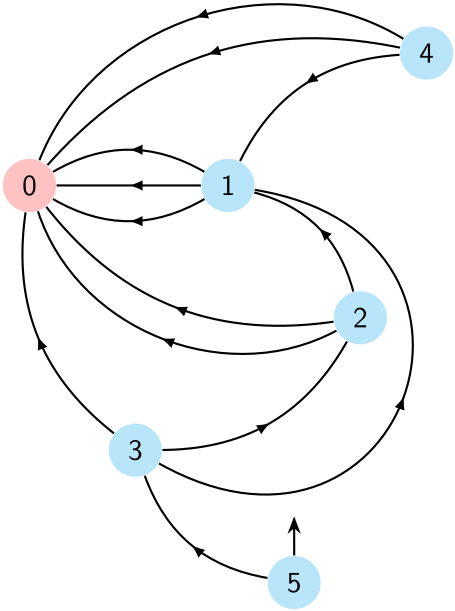
$s =$	Deg.
0	8
1	6
2	4
3	4
4	3
5	1

Time 5, Edge #14



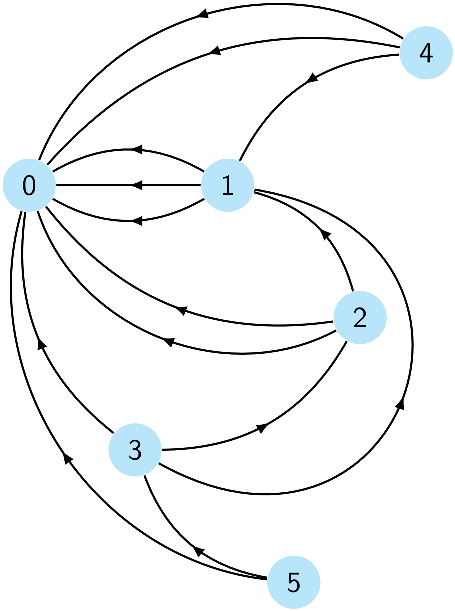
$s =$	Deg.	Pr(choose s) $\propto$
0	8	$8 + \delta(5)$
1	6	$6 + \delta(5)$
2	4	$4 + \delta(5)$
3	4	$4 + \delta(5)$
4	3	$3 + \delta(5)$
5	1	

Time 5, Edge #14



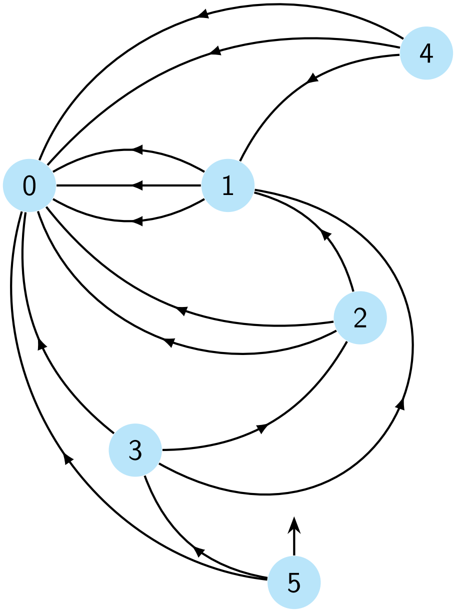
$s =$	Deg.	Pr(choose s) $\propto$
0	8	$8 + \delta(5)$
1	6	$6 + \delta(5)$
2	4	$4 + \delta(5)$
3	4	$4 + \delta(5)$
4	3	$3 + \delta(5)$
5	1	

Time 5, Edge #14



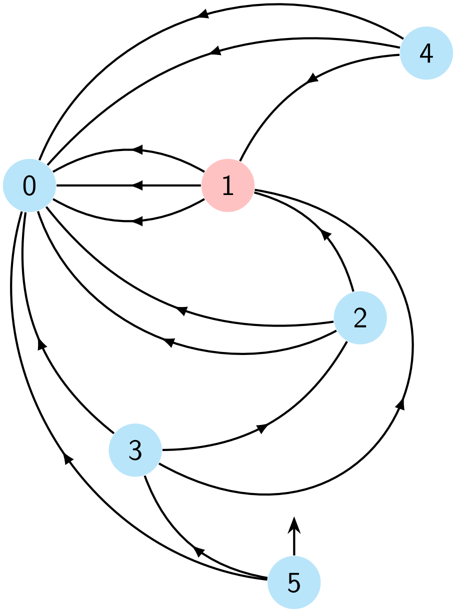
$s =$	Deg.
0	9
1	6
2	4
3	4
4	3
5	2

Time 5, Edge #15



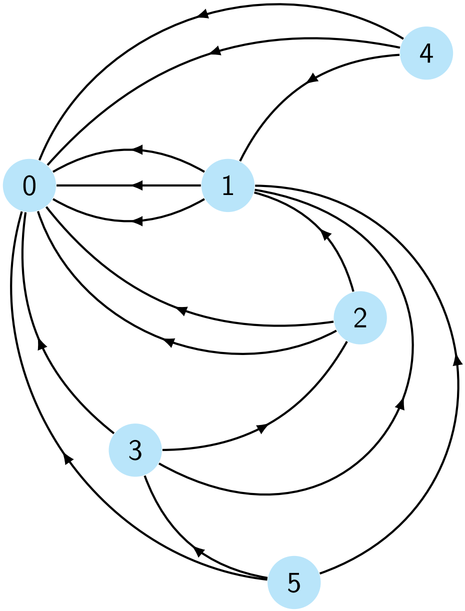
$s =$	Deg.	Pr(choose s) $\propto$
0	9	$9 + \delta(5)$
1	6	$6 + \delta(5)$
2	4	$4 + \delta(5)$
3	4	$4 + \delta(5)$
4	3	$3 + \delta(5)$
5	2	

Time 5, Edge #15



$s =$	Deg.	Pr(choose s) $\propto$
0	9	$9 + \delta(5)$
1	6	$6 + \delta(5)$
2	4	$4 + \delta(5)$
3	4	$4 + \delta(5)$
4	3	$3 + \delta(5)$
5	2	

Time 5, Edge #15



$s =$	Deg.
0	9
1	7
2	4
3	4
4	3
5	3

G_5

Preferential attachment (multi)graph

We build $(G_0, G_1, G_2, \dots)$ sequentially. At time $t \geq 1$ the incoming vertex t "sends" m edges to previous vertices $\{0, \dots, t-1\}$.

After vertex t has sent $i-1$ edges, the probability that the i-th edge choose vertex $s \in \{0, \dots, t-1\}$ is

$$\propto \deg_{G_{t-1}}(s) + \#(\text{edges between } t \text{ and } s) + \delta(t)$$

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\end{aligned}$$

$\Longrightarrow$ Edges of vertex t follow a Pólya urn conditional on G_{t-1}.

Basic properties

- The PA process generates almost-surely *Directed Acyclic (multi)Graphs* (DAG) ;
- Richer-get-richer phenomenon ;
- Can generate graphs with power-law degree distributions ;
- Allegedly model the dynamic of "real networks"
- Dynamic of the graph controlled by the function $\delta : \mathbb{N} \to (-m, +\infty)$ (constant or piecewise constant in the sequel)
  - Constant : no changepoint ;
  - Piecewise constant : existence of changepoints

Hypothesis testing problem

The change-point detection can be seen as a hypothesis testing problem :

$$\begin{aligned} &(\mathrm{H}_0) \quad (\exists \delta_0 > -m) \quad \delta(t) = \delta_0 \\ &(\mathrm{H}_1) \quad (\exists \delta_1 \neq \delta_0 > -m) \quad \delta(t) = \delta_0 \mathbf{1}_{t \le \tau_n} + \delta_1 \mathbf{1}_{t > \tau_n} \end{aligned}$$

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Prior work focuses mainly on the case $m = 1$ and early detection where the change happens at a linear time $\mathrm{O}(n)$ or even $\mathrm{o}(n)$. We focus here on the case of late change-point detection, i.e.

$$\tau_n = n - \Delta_n \text{ and } \Delta_n = o(n)$$

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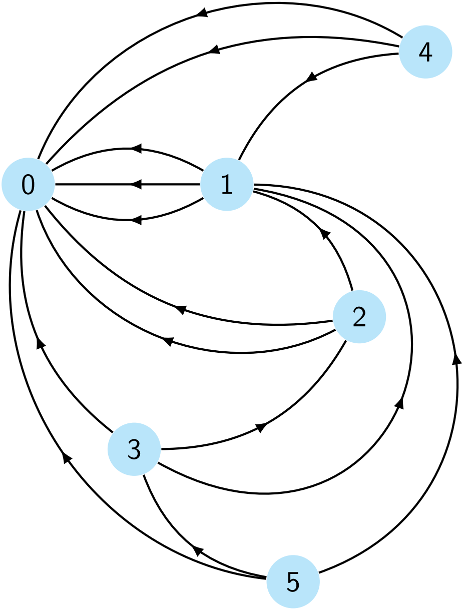
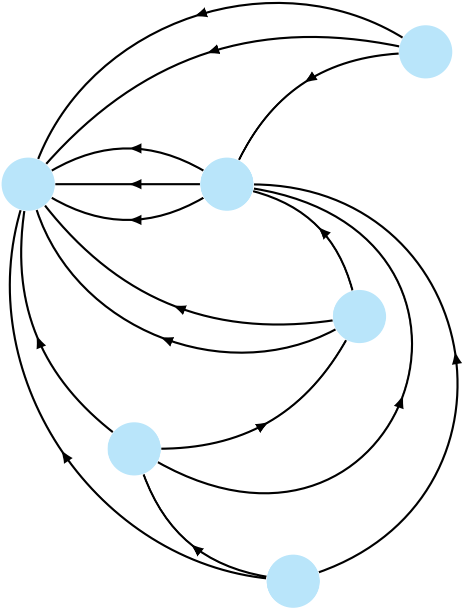
That is we want to understand the minimal time needed to detect the change-point after it occurs.

Not the whole story though ! We are interested in the detection of the change-point based only on the observation of the **unlabeled graph**.


0
1
2
3
4
5



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Unlabeled directed graphs

- For two (labeled) graphs $g, g' \in \mathcal{G}$, $g \cong g'$ iff g and g' are isomorphic ;
- $s(g) \equiv [g]_{\cong}$ is the *structure* of the graph ; namely the isomorphism class of g ;
- $\mathcal{U}_n = \{s(g) \; : \; g \in \mathcal{G}_n\}$ the set of *unlabeled directed multigraphs* on $n + 1$ vertices.

Formal statement of the problem

At this stage, we have two distributions over the set $\mathcal{G}_n$ of labeled directed graphs [aka. laws of G_n] :

$$P_0^n \text{ when } \delta(t) = \delta_0$$
$$P_1^n \text{ when } \delta(t) = \delta_0 \mathbf{1}_{t \leq \tau_n} + \delta_1 \mathbf{1}_{t > \tau_n}$$

which induce through s two distributions over the set of unlabeled directed graphs $\mathcal{U}_n$ [aka. laws of $U_n = s(G_n)$]

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We consider the hypothesis testing problem :

$$H_0 : P = P_0^{n,\cong} \qquad H_1 : P = P_1^{n,\cong}.$$

Bet et al. 2023's result

They consider $\Delta_n = \lfloor cn^\gamma \rfloor$ with $c > 0$ and $\gamma \in (0, 1)$.

Theorem (Bet et al. 2023)

There exists a sequence of test $(\psi_n)_{n\geq 0}$ with $\psi_n : \mathcal{U}_n \to [0, 1]$ such that

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- $\lim_n E_{P_1^{n,\cong}}[\psi_n(U_n)] = 0$ *when* $\gamma < 1/2.$

Conjecture (Bet et al. 2023)

When $\gamma \leq 1/2$ *there is no sequence* $(\phi_n)_{n\geq 0}$, $\phi_n : \mathcal{U}_n \to [0,1]$, *such that*

$$\lim_n E_{P_0^{n,\cong}}[\phi_n(U_n)] = 0, \quad \text{and,} \quad \lim_n E_{P_1^{n,\cong}}[\phi_n(U_n)] = 1.$$

Our result

Theorem (EG, I. Kaddouri, Z. Naulet 2024)

Recall $\tau_n = n - \Delta_n$.
If $\delta_0 > 0$ *and* $\Delta_n = o(n^{1/3})$ *or if* $\delta_0 = 0$ *and* $\Delta_n = o\left(\frac{n^{1/3}}{\log n}\right)$, *then for every sequence* $(\phi_n)_{n\geq 0}$ *of tests* $\phi_n : \mathcal{U}_n \to [0,1]$

$$\lim_n E_{P_0^{n,\cong}}[\phi_n(U_n)] = 0 \implies \lim_n E_{P_1^{n,\cong}}[\phi_n(U_n)] = 0$$

Another result with some interest

The problem is quite different if we observe the labeled graph

Theorem (EG, I. Kaddouri, Z. Naulet 2024)

Suppose $\tau_n \to +\infty$ *and* $\Delta_n \to \infty$*. Then, there is a sequence* $(\psi_n)_{n \geq 0}$, $\psi_n : \mathcal{G}_n \to [0, 1]$*, such that*

$$\lim_n E_{P_0^n}[\psi_n(G_n)] = 0, \quad \text{and}, \quad \lim_n E_{P_1^n}[\psi_n(G_n)] = 1.$$

When $\limsup_{n \to +\infty} \Delta_n < \infty$*, detection of the change is not possible.*

About the difficulty of the problem

Traditional path to simultaneously prove Theorem 1 and the conjecture : require to understand the *likelihood ratio*

$$\frac{\mathrm{d}P_1^{n,\cong}}{\mathrm{d}P_0^{n,\cong}}(U_n)$$

under $U_n \sim P_0^{n,\cong}$ (enough for conjecture) and $U_n \sim P_1^{n,\cong}$ (for the Theorem).

- Neyman-Pearson's Lemma guarantees that $\mathbf{1}\Big(\frac{\mathrm{d}P_1^{n,\cong}}{\mathrm{d}P_0^{n,\cong}}(U_n) > k\Big)$ is uniformly most powerful amongst all the tests of its size ;
- Our impossibility Theorem is equivalent to the statement

$$P_0^{n,\simeq}(A_n) \to 0 \implies P_1^{n,\simeq}(A_n) \to 0$$

usually called contiguity and written $P_1^{n,\simeq} \lhd P_0^{n,\simeq}$ (well-studied by Le Cam).

So let's try to do this.

Let μ_n (respectively λ_n) be the counting measure on $\mathcal{G}_n$ (resp. $\mathcal{U}_n$). Then,

$$\frac{\mathrm{d}P_j^{n,\cong}}{\mathrm{d}\lambda_n}(u) = \sum_{\substack{g \in u \\ V(g)=[n]}} \frac{\mathrm{d}P_j^n}{\mathrm{d}\mu_n}(g)$$

Difficult to understand ! ! ! Why ?

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Difficult to understand ! ! ! Why ?

Consider the two isomorphic graphs :

$g =$ 

$g' =$ 

$$\frac{\mathrm{d}P_j^3}{\mathrm{d}\mu_3}(g) > 0 \quad \text{but} \quad \frac{\mathrm{d}P_j^3}{\mathrm{d}\mu_3}(g') = 0.$$

$\Longrightarrow$ makes the above sum tricky to analyze.

$\Longrightarrow$ need of another strategy.

1. Introduction : what is the PA model ? what is a changepoint ? Statement of main results and the **conjecture**

2. Feasibility : how to detect a late changepoint.

3. Impossibility : when is it impossible to detect a late change-point ?

Strategy from Bet et al (2023).

To prove the feasibility, it is enough to build a test. The main intuitions are the following :

- Under H_0 the degree statistics $(N_m(U_n), N_{m+1}(U_n), \dots)$ where

  $$N_k(U_n) := \text{Number of vertices of degree } k \text{ in } U_n$$

  is a sufficient statistic for the model (not under H_1 though). Furthermore, under both P_0^n and P_1^n

  $$(N_m(s(G_n)), N_{m+1}(s(G_n)), \dots) \overset{law}{=} (N_m(G_n), N_{m+1}(G_n), \dots)$$

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For now : we assume δ_0 is known.

So typically we need to understand the behaviour of $N_m(U_n)$ and show that it differs under H_0 and H_1

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More precisely, defining

$$p_m(\delta_0) := \lim_{n\to\infty} \frac{E_0(N_m(U_n))}{n}$$

they show that the test statistic

$$T_n(U_n) := N_m(U_n) - np_m(\delta_0)$$

satisfies

1. $T_n(U_n) = O_p(\sqrt{n})$ under $U_n \sim P_0^{n,\cong}$;
2. $T_n(U_n) = C(p_m(\delta_1) - p_m(\delta_0))n^\gamma + O_p(\sqrt{n})$ under $U_n \sim P_1^{n,\cong}$.

When δ_0 is unknown

In this case, one can simply consider an estimator $\hat{\delta}_n(U_n)$ of δ_0 and consider the plugin test

$$Q(U_n) = N_m(n) - np_m(\hat{\delta}_n(U_n))$$

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But what estimator ?

Interestingly, the MLE under H_0 is easy to compute (Gao and van der Vaart, 2017). Consider a labeled graph g_n in the support of P_0^n. Then

$$\frac{\mathrm{d}P_0^n}{\mathrm{d}\mu_n}(g_n) = \frac{\prod_{k>m}\left[\prod_{j=m}^{k}(j+\delta_0)\right]^{N_k(g_n)}}{\prod_{t=2}^{n}\prod_{i=1}^{m}(2m(t-1)+(i-1)+t\delta_0)}$$

This only depends on $s(g_n)$! !

Deduce that,

$$\frac{\mathrm{d}P_0^{n,\cong}}{\mathrm{d}\lambda_n}(u_n) \propto \frac{\prod_{k>m}\left[\prod_{j=m}^{k}(j+\delta_0)\right]^{N_k(u_n)}}{\prod_{t=2}^{n}\prod_{i=1}^{m}(2m(t-1)+(i-1)+t\delta_0)},$$

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$\rightarrow$ analyzis of the MLE.

$$\begin{aligned} Q(U_n) &= N_m(n) - np_m(\hat{\delta}_n(U_n)) \\ &= \underbrace{N_m(n) - np_m(\delta_0)}_{(1)} + \underbrace{n(p_m(\hat{\delta}_n) - p_m(\delta_0))}_{(2)} \end{aligned}$$

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Under H_0, $(1) = \mathrm{O}_{P_0^n}\left(\sqrt{n}\right)$ and $(2) = \mathrm{O}_{P_0^n}\left(\sqrt{n}\log(n)\right)$.

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Under H_1 when $\gamma > 1/2$, $(1) \asymp n^{\gamma}$ and $(2) \asymp n^{\gamma}$ and $(1)+(2) \asymp n^{\gamma}$.

Recall our result

Theorem (EG, I. Kaddouri, Z. Naulet 2024)

Recall $\tau_n = n - \Delta_n$.
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$$\lim_n E_{P_0^{n,\cong}}[\phi_n(U_n)] = 0 \implies \lim_n E_{P_1^{n,\cong}}[\phi_n(U_n)] = 0$$

Le Cam's theory : equivalent to show that $P_1^{n,\cong} \lhd P_0^{n,\cong}$.
Too difficult to establish.

Reduction

Usual approach : show that an easier problem is not feasible.
Most trivial attempt : show that the change-point cannot be detected from the labeled graph G_n itself... unfortunately :

Theorem (EG, I. Kaddouri, Z. Naulet 2024)

Suppose $\tau_n \to +\infty$ *and* $\Delta_n \to \infty$. *Then, there is a sequence* $(\psi_n)_{n \geq 0}$, $\psi_n : \mathcal{G}_n \to [0,1]$, *such that*

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Using MLE (and standard martingale arguments as in Gao et al. 2017).

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Using MLE (and standard martingale arguments as in Gao et al. 2017).

We need to find a problem that is intermediate between observing $s(G_n)$ and G_n

Simpler problem : consider the observation of $\pi(G_n)$ where π is a random permutation.

- $\mathbb{P}_0^n$ (respectively $\mathbb{P}_1^n$) joint distribution of (π, G_n) under H_0 (resp. H_1).
- $P_0^{n,\pi}$ distributions of $\pi(G_n)$ under H_0 ; idem $P_1^{n,\pi}$;
- P_0^n marginal distribution of G_n under H_0 ; idem P_1^n ;

We have to construct $\mathbb{P}_j^n$ such that :

A. It is easier to test $\{H_0, H_1\}$ based on the observation of $\pi(G_n)$ rather than $U_n = s(G_n)$:

B. We can prove that $P_1^{n,\pi} \lhd P_0^{n,\pi}$.

Remark : to prove B, it is convenient to build the permutation π conditional to G_n ; *ie.* π and G_n are not independent and it is unclear if A is true.

A first lemma : hierarchy of difficulties

Lemma

Suppose the law of $\pi \mid G_n$ *is the same under* H_0 *and* H_1 *and consider the following statements :*

1. *For every sequence* (ϕ_n) *of* G_n*-measurable tests,* $\lim_n E_{\mathbb{P}_0^n}(\phi_n) = 0$ *implies* $\lim_n E_{\mathbb{P}_1^n}(\phi_n) = 0$ *;*
2. *For every sequence* (ϕ_n) *of* $\pi(G_n)$*-measurable tests,* $\lim_n E_{\mathbb{P}_0^n}(\phi_n) = 0$ *implies* $\lim_n E_{\mathbb{P}_1^n}(\phi_n) = 0$ *;*
3. *For every sequence* (ϕ_n) *of* $s(G_n)$*-measurable tests,* $\lim_n E_{\mathbb{P}_0^n}(\phi_n) = 0$ *implies* $\lim_n E_{\mathbb{P}_1^n}(\phi_n) = 0$*.*

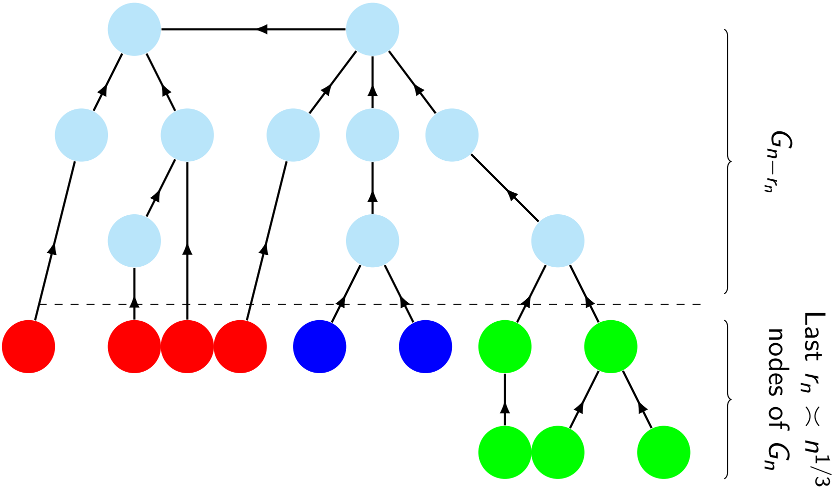
Then $1 \implies 2 \implies 3$*.*

So now we shall investigate permutations :

- complicated enough to enable contiguity.
- tractable enough to enable computations.
- built as prescribed by the previous lemma.

Contiguity : we use a second moment bound. For any sequence $(A_n)_{n\geq 1}$ of events and with $H_n = \pi(G_n)$

$$
\begin{aligned}
P_1^{n,\pi}(A_n) &= P_1^{n,\pi}(A_n \cap B_n^c) + P_1^{n,\pi}(A_n \cap B_n) \\
&\leq P_1^{n,\pi}(B_n^c) + E_{P_1^{n,\pi}}\left(\mathbf{1}_{A_n\cap B_n}(H_n)\right) \\
&= P_1^{n,\pi}(B_n^c) + E_{P_0^{n,\pi}}\left(\mathbf{1}_{A_n\cap B_n}(H_n)\frac{\mathrm{d}P_1^{n,\pi}}{\mathrm{d}P_0^{n,\pi}}(H_n)\right) \\
&\leq P_1^{n,\pi}(B_n^c) + P_0^{n,\pi}(A_n)^{1/2}\left(E_{\mathbb{P}_0^n}\left[\left(\frac{\mathrm{d}P_1^{n,\pi}}{\mathrm{d}P_0^{n,\pi}}(H_n)\right)^2 \mathbf{1}_{B_n}(H_n)\right]\right)^{1/2}
\end{aligned}
$$



- Vertices in red : Swap their labels with "compatible" vertices in $[n - r_n]$;
- Vertices in blue or green : keep their labels

B_n is the event that no blue or green nodes exists.

- Advantage : this event has probability $1 + o(1)$ as long as $r_n = o(n^{1/3})$ and on this event the likelihood ratio $\frac{\mathrm{d}P_1^{n,\pi}}{\mathrm{d}P_0^{n,\pi}}(\pi(G_n))$ behave nicely under $\mathbb{P}_0^n$;
- Advantage bis : This random permutation leaves invariant P_0^n : *ie.* $\pi(G_n) \sim P_0^n$ when $(\pi, G_n) \sim \mathbb{P}_0^n$. This makes the study under $\mathbb{P}_0^n$ convenient.
- Downside : this event has probability going to zero as long as $r_n \gg n^{1/3}$; so this trick cannot work to fully establish the conjecture...

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- What next ?

Thank you for your attention !