Notes for the *Winter School at Canazei*

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1. Welfarism

The set of alternatives X contains at least three elements A utility profile is an n-tuple $U = (U_1, \ldots, U_n)$, where $U_i: X \to \mathcal{R}$ is the utility function of individual $i \in \{1, \ldots, n\}$. The set of all possible utility profiles is $\mathcal{U}$ and we write $U(x) = (U_1(x), \ldots, U_n(x))$ for all $x \in X$ and for all $U \in \mathcal{U}$.

Social and individual non-welfare information for the fixed population $\{1, \ldots, n\}$ is described by an $(n+1)$-tuple $K = (K_0, K_1, \ldots, K_n)$, where $K_0: X \to \mathcal{S}_0$ and $K_i: X \to \mathcal{S}_i$ for all

$i \in \{1, \ldots, n\}$. For $x \in X$, $K_0(x)$ is social non-welfare information in alternative x and, for all $i \in \{1, \ldots, n\}$, $K_i(x)$ is non-welfare information for person i in alternative x. $\mathcal{S}_0 \neq \emptyset$ and $\mathcal{S}_i \neq \emptyset$ are the sets of possible values of non-welfare information for society and individual i respectively. The set of all possible profiles of non-welfare information is $\mathcal{K}$ and, for all $x \in X$ and for all $K \in \mathcal{K}$, $K(x) = (K_0(x), K_1(x), \ldots, K_n(x))$.

The set of all orderings on X is denoted by $\mathcal{O}$. A social-evaluation functional is a mapping $F{:}\,\mathcal{D} \to \mathcal{O}$, where $\mathcal{D} \subseteq \mathcal{U} \times \mathcal{K}$ is the domain of F, assumed to be non-empty. For convenience, we use the notation $\Upsilon = (U, K)$ and $R_\Upsilon = F(\Upsilon)$ for all $\Upsilon \in \mathcal{D}$. Furthermore, we define

$$\Upsilon(x) = (U(x), K(x)) \qquad (1.1)$$

for all $x \in X$. The asymmetric factor and the symmetric factor of R_Υ are denoted by P_Υ and I_Υ.

Minimal Individual Goodness: For all $x, y \in X$ and for all $\Upsilon \in \mathcal{D}$, if $xP_\Upsilon y$, then there exists $j \in \{1, \ldots, n\}$ such that $U_j(x) > U_j(y)$.

Pareto Indifference: For all $x, y \in X$ and for all $\Upsilon \in \mathcal{D}$, if $U(x) = U(y)$, then $xI_\Upsilon y$.

Pareto Weak Preference: For all $x, y \in X$ and for all $\Upsilon \in \mathcal{D}$, if $U(x) > U(y)$, then $xR_\Upsilon y$.

Theorem A: *F satisfies minimal individual goodness if and only if F satisfies Pareto indifference and Pareto weak preference.*

Proof. Suppose F satisfies minimal individual goodness. We first prove by contradiction that Pareto indifference is satisfied. Suppose not. Then there exist $x, y \in X$ and $\Upsilon \in \mathcal{D}$ such that $U(x) = U(y)$ and not $xI_\Upsilon y$. Because R_Υ is complete, we must have either $xP_\Upsilon y$ or $yP_\Upsilon x$. In each case, we obtain a contradiction to minimal individual goodness.

Now suppose F violates Pareto weak preference. Then there exist $x, y \in X$ and $\Upsilon \in \mathcal{D}$ such that $U(x) > U(y)$ and not $xR_\Upsilon y$. By the completeness of R_Υ, we must have $yP_\Upsilon x$, again contradicting minimal individual goodness.

Finally, suppose F satisfies Pareto indifference and Pareto weak preference but violates minimal individual goodness. Then there exist $x, y \in X$ and $\Upsilon \in \mathcal{D}$ such that $xP_\Upsilon y$ and $U(y) \geq U(x)$. If $U(y) = U(x)$, we obtain a contradiction to Pareto indifference, and if there exists $j \in \{1, \ldots, n\}$ such that $U_j(y) > U_j(x)$, we obtain a contradiction to Pareto weak preference. ■

Unlimited Domain: $\mathcal{D} = \mathcal{U} \times \mathcal{K}$.

Binary Independence of Irrelevant Alternatives: For all $x, y \in X$ and for all $\Upsilon, \bar{\Upsilon} \in \mathcal{D}$, if $\Upsilon(x) = \bar{\Upsilon}(x)$ and $\Upsilon(y) = \bar{\Upsilon}(y)$, then

$$xR_\Upsilon y \Leftrightarrow xR_{\bar{\Upsilon}} y.$$

Theorem B: *If F satisfies unlimited domain, Pareto indifference and binary independence of irrelevant alternatives, then, for all $x, y \in X$ and for all $\Upsilon, \bar{\Upsilon} \in \mathcal{D}$ such that $U(x) = \bar{U}(x)$ and $U(y) = \bar{U}(y)$,*

$$xR_\Upsilon y \Leftrightarrow xR_{\bar{\Upsilon}} y. \qquad (1.2)$$

Proof. Let $x, y \in X$ and $\Upsilon, \bar{\Upsilon} \in \mathcal{D}$ be such that $U(x) = \bar{U}(x)$ and $U(y) = \bar{U}(y)$. Let $u = U(x) = \bar{U}(x)$, $v = U(y) = \bar{U}(y)$, $k = K(x)$, $\ell = K(y)$, $\bar{k} = \bar{K}(x)$ and $\bar{\ell} = \bar{K}(y)$. Because X contains at least three alternatives, there exists $z \in X \setminus \{x, y\}$. By unlimited domain, we can find profiles Υ_a, Υ_b, Υ_c and Υ_d with the following properties. Let $\Upsilon_a(x) = (u, k)$, $\Upsilon_a(y) = (v, \ell)$, $\Upsilon_a(z) = (v, \bar{\ell})$, $\Upsilon_b(x) = (u, k)$, $\Upsilon_b(y) = (v, \bar{\ell})$, $\Upsilon_b(z) = (v, \bar{\ell})$, $\Upsilon_c(x) = (u, k)$, $\Upsilon_c(y) = (v, \bar{\ell})$, $\Upsilon_c(z) = (u, \bar{k})$, $\Upsilon_d(x) = (u, \bar{k})$, $\Upsilon_d(y) = (v, \bar{\ell})$ and $\Upsilon_d(z) = (u, \bar{k})$.

By binary independence of irrelevant alternatives, we have

$$xR_{\Upsilon}y \Leftrightarrow xR_{\Upsilon_a}y.$$

By Pareto indifference, $yI_{\Upsilon_a}z$ and it follows that

$$xR_{\Upsilon_a}y \Leftrightarrow xR_{\Upsilon_a}z.$$

Using binary independence again, we obtain

$$xR_{\Upsilon_a}z \Leftrightarrow xR_{\Upsilon_b}z.$$

By Pareto indifference, $zI_{\Upsilon_b}y$ and, therefore,

$$xR_{\Upsilon_b}z \Leftrightarrow xR_{\Upsilon_b}y.$$

Now binary independence implies

$$xR_{\Upsilon_b}y \Leftrightarrow xR_{\Upsilon_c}y.$$

By Pareto indifference, $xI_{\Upsilon_c}z$ and it follows that

$$xR_{\Upsilon_c}y \Leftrightarrow zR_{\Upsilon_c}y.$$

Using binary independence again, we obtain

$$zR_{\Upsilon_c}y \Leftrightarrow zR_{\Upsilon_d}y.$$

By Pareto indifference, $zI_{\Upsilon_d}x$ and it follows that

$$zR_{\Upsilon_d}y \Leftrightarrow xR_{\Upsilon_d}y.$$

Using binary independence once more, we obtain

$$xR_{\Upsilon_d}y \Leftrightarrow xR_{\bar{\Upsilon}}y.$$

Combining the above equivalences, (1.2) results. ■

Strong Neutrality: For all $x, y, z, w \in X$ and for all $\Upsilon, \bar{\Upsilon} \in \mathcal{D}$, if $U(x) = \bar{U}(z)$ and $U(y) = \bar{U}(w)$, then

$$xR_{\Upsilon}y \Leftrightarrow zR_{\bar{\Upsilon}}w.$$

Note that, in the formulation of strong neutrality, non-welfare information is allowed to be different in x and z and in y and w. In contrast, this is not the case in the axiom binary independence of irrelevant alternatives. Thus, adding Pareto indifference to the independence condition produces a remarkably strong result by eliminating the possible influence of non-welfare information altogether.

Theorem C: *Suppose F satisfies unlimited domain. F satisfies Pareto indifference and binary independence of irrelevant alternatives if and only if F satisfies strong neutrality.*

Proof. First, suppose that F satisfies strong neutrality. That binary independence of irrelevant alternatives is satisfied follows from setting $x = z$, $y = w$, $K(x) = \bar{K}(x)$ and $K(y) = \bar{K}(y)$ in the definition of strong neutrality. To show that Pareto indifference is implied, let $U = \bar{U}$ and $y = z = w$. Strong neutrality implies that $xR_{\Upsilon}y$ if and only if

$yR_\Upsilon y$ whenever $U(x) = U(y)$. Because R_Υ is reflexive, this implies $xI_\Upsilon y$.

Now suppose that F satisfies unlimited domain, Pareto indifference and binary independence of irrelevant alternatives. By Theorem B, we know that non-welfare information is irrelevant. Consider two profiles $\Upsilon, \bar{\Upsilon} \in \mathcal{D}$ and four (not necessarily distinct) alternatives $x, y, z, w \in X$ such that $U(x) = \bar{U}(z) = u$ and $U(y) = \bar{U}(w) = v$.

By unlimited domain, there exist profiles $\Upsilon_a, \Upsilon_b, \Upsilon_c, \Upsilon_d \in \mathcal{D}$ such that $U_a(x) = u$, $U_a(y) = v$, $U_a(w) = v$, $U_b(x) = u$, $U_b(y) = v$, $U_b(w) = v$, $U_c(x) = u$, $U_c(y) = v$, $U_c(z) = u$, $U_d(y) = v$, $U_d(z) = u$ and $U_d(w) = v$.

By binary independence of irrelevant alternatives,

$$xR_\Upsilon y \Leftrightarrow xR_{\Upsilon_a} y.$$

By Pareto indifference, $yI_{\Upsilon_a}w$ and, therefore,

$$xR_{\Upsilon_a}y \Leftrightarrow xR_{\Upsilon_a}w.$$

Using binary independence of irrelevant alternatives again, we obtain

$$xR_{\Upsilon_a}w \Leftrightarrow xR_{\Upsilon_b}w.$$

By Pareto indifference, $yI_{\Upsilon_b}w$ and, therefore,

$$xR_{\Upsilon_b}w \Leftrightarrow xR_{\Upsilon_b}y.$$

By binary independence of irrelevant alternatives,

$$xR_{\Upsilon_b}y \Leftrightarrow xR_{\Upsilon_c}y.$$

By Pareto indifference, $xI_{\Upsilon_c}z$ and, therefore,

$$xR_{\Upsilon_c}y \Leftrightarrow zR_{\Upsilon_c}y.$$

By binary independence of irrelevant alternatives,

$$zR_{\Upsilon_c}y \Leftrightarrow zR_{\Upsilon_d}y.$$

By Pareto indifference, $yI_{\Upsilon_d}w$ and, therefore,

$$zR_{\Upsilon_d}y \Leftrightarrow zR_{\Upsilon_d}w$$

and, using binary independence of irrelevant alternatives once more, we obtain

$$zR_{\Upsilon_d}w \Leftrightarrow zR_{\bar{\Upsilon}}w.$$

Combining the above equivalences, we obtain

$$xR_{\Upsilon}y \Leftrightarrow zR_{\bar{\Upsilon}}w,$$

and strong neutrality is satisfied. ■

Theorem D: *Suppose F satisfies unlimited domain. F satisfies Pareto indifference and binary independence of irrelevant alternatives if and only if there exists a social-evaluation ordering R on $\mathcal{R}$ such that, for all $x, y \in X$ and for all $\Upsilon \in \mathcal{D}$,*

$$xR_{\Upsilon}y \Leftrightarrow U(x)RU(y). \tag{1.3}$$

Proof. Clearly, the existence of a social-evaluation ordering R such that, for all $x, y \in X$ and for all $\Upsilon \in \mathcal{D}$, (1.3) is satisfied implies Pareto indifference and binary independence of irrelevant alternatives.

Now suppose F satisfies unlimited domain, Pareto indifference and binary independence of irrelevant alternatives. By Theorem C, F satisfies strong neutrality. We complete the proof by constructing the social-evaluation ordering R. For all $u, v \in \mathcal{R}$, let uRv if and only if there exist a profile $\Upsilon \in \mathcal{D}$ and two alternatives $x, y \in X$ such that $U(x) = u$, $U(y) = v$ and $xR_\Upsilon y$ (the existence of the profile Υ and the alternatives x and y is guaranteed by unlimited domain). Strong neutrality implies that non-welfare information is irrelevant and that the ranking of any two utility vectors u and v

does not depend on the profile Υ or on the alternatives x and y used to generate u and v. Therefore, R is well-defined. That R is reflexive and complete follows immediately because R_Υ is reflexive and complete for all $\Upsilon \in \mathcal{D}$. We have to show that R is transitive. Suppose $u, v, q \in \mathcal{R}$ are such that uRv and vRq. By unlimited domain and the assumption that X contains at least three alternatives, there exist a profile $\Upsilon \in \mathcal{D}$ and three alternatives $x, y, z \in X$ such that $U(x) = u$, $U(y) = v$ and $U(z) = q$. Because $U(x)RU(y)$ and $U(y)RU(z)$, it follows that $xR_\Upsilon y$ and $yR_\Upsilon z$ by definition of R. Because R_Υ is transitive, we have $xR_\Upsilon z$. Hence, $U(x)RU(z)$ or, equivalently, uRq. ■

2. Generalized Utilitarianism

Suppose that the ordering R satisfies:

Same-People Anonymity: For all $u \in \mathcal{R}^n$ and for all bijections $\rho\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$,

$$uI\big(u_{\rho(1)}, \ldots, u_{\rho(n)}\big);$$

Strong Pareto: For all $u, v \in \mathcal{R}^n$, if $u > v$, then uPv,

Continuity: For all $u \in \mathcal{R}^n$, the sets $\{v \in \mathcal{R}^n \mid vRu\}$ and $\{v \in \mathcal{R}^n \mid uRv\}$ are closed in $\mathcal{R}^n$

and

Same-People Independence: For all M such that $\emptyset \neq M \subset \{1, \ldots, n\}$ and for all

$u, v, \bar{u}, \bar{v} \in \mathcal{R}^n$, if $[u_i = v_i$ and $\bar{u}_i = \bar{v}_i]$ for all $i \in M$ and $[u_j = \bar{u}_j$ and $v_j = \bar{v}_j]$ for all $j \in \{1, \ldots, n\} \setminus M$, then

$$uRv \Leftrightarrow \bar{u}R\bar{v}.$$

Theorem E: *Suppose $n \geq 3$. R satisfies continuity, same-people anonymity, strong Pareto and same-people independence if and only if R is generalized-utilitarian, that is,*

$$uRv \Leftrightarrow \sum_{i=1}^{n} g(u_i) \geq \sum_{i=1}^{n} g(v_i) \qquad (2.1)$$

where g is increasing and $g(0) = 0$.

Proof. That generalized utilitarianism satisfies the required axioms is straightforward to verify. Conversely, let $n \geq 3$ and suppose that R^n satisfies continuity, same-people anonymity, strong Pareto and same-people independence. Applying Debreu's [1959, pp. 56–59] representation theorem, there exists a continuous function $f\colon \mathcal{R}^n \to \mathcal{R}$ such that, for all $u, v \in \mathcal{R}^n$,

$$uRv \Leftrightarrow f(u) \geq f(v).$$

By strong Pareto, f is increasing in all arguments, and same-people anonymity implies that f is symmetric. In addition, same-people independence implies that $\{1, \ldots, n\} \setminus M$ is separable from its (non-empty) complement M for any choice of M such that $\emptyset \neq M \subset \{1, \ldots, n\}$. Gorman's [1968] theorem on overlapping separable sets of variables implies that

f is additively separable. Therefore, there exist continuous and increasing functions $H: \mathcal{R} \to \mathcal{R}$ and $g_i: \mathcal{R} \to \mathcal{R}$ for all $i \in \{1, \ldots, n\}$ such that

$$f(u) = H\left(\sum_{i=1}^{n} g_i(u_i) \right)$$

for all $u \in \mathcal{R}^n$. Because f is symmetric, each g_i can be chosen to be independent of i, and we define $g_i = g$ for all $i \in \{1, \ldots, n\}$. Therefore, because f is a representation of R,

$$\begin{aligned} uRv &\Leftrightarrow H\left(\sum_{i=1}^{n} g(u_i) \right) \geq H\left(\sum_{i=1}^{n} g(v_i) \right) \\ &\Leftrightarrow \sum_{i=1}^{n} g(u_i) \geq \sum_{i=1}^{n} g(v_i) \end{aligned}$$

for all $u, v \in \mathcal{R}^n$. Without loss of generality, g can be chosen so that $g(0) = 0$. ■

3. Intertemporal Welfare and Non Welfare Characteristics

Because our objective is to examine the intertemporal aspects of social evaluation, we focus on birth dates and lengths of life as the non-welfare information that may be of relevance. For each $i \in \{1, \ldots, n\}$, $S_i\colon X \to \mathcal{Z}_+$ assigns the period just before i is born to each alternative. Analogously, $L_i\colon X \to \{1, \ldots, \bar{L}\}$ is a function that specifies i's lifetime for each alternative. Thus, in alternative $x \in X$, i is alive in periods $S_i(x) + 1, \ldots, S_i(x) + L_i(x)$. A period-before-birth-date profile is an n-tuple $S = (S_1, \ldots, S_n)$ and the set of all logically possible period-before-birth-date profiles is $\mathcal{S}$. Analogously, a length-of-life profile is an n-tuple $L = (L_1, \ldots, L_n)$

and the set of all logically possible length-of-life profiles is $\mathcal{L}$. Furthermore, we define $S(x) = (S_1(x), \ldots, S_n(x))$ and $L(x) = (L_1(x), \ldots, L_n(x))$ for all $x \in X$.

An information profile collects welfare information and non-welfare information in a vector $\Upsilon = (U, S, L) \in \mathcal{U} \times \mathcal{S} \times \mathcal{L}$. For $x \in X$, we write $\Upsilon(x) = (U(x), S(x), L(x))$. We define $\Omega = \mathcal{R} \times \mathcal{Z}_+ \times \{1, \ldots, \bar{L}\}$, and the set of possible compound vectors (u, s, ℓ) of utility vectors, vectors of periods before birth and vectors of lengths of life is $\Omega^n = \mathcal{R}^n \times \mathcal{Z}_+^n \times \{1, \ldots, \bar{L}\}^n$.

A social-evaluation functional is a mapping $F: \mathcal{D} \to \mathcal{O}$ where $\emptyset \neq \mathcal{D} \subseteq \mathcal{U} \times \mathcal{S} \times \mathcal{L}$ is the domain of F. We write $R_\Upsilon = F(\Upsilon)$ for all $\Upsilon \in \mathcal{D}$. The asymmetric and symmetric factors of R_Υ are P_Υ and I_Υ.

Unlimited domain: $\mathcal{D} = \mathcal{U} \times \mathcal{S} \times \mathcal{L}$.

Conditional Pareto indifference: For all $x, y \in X$ and for all $\Upsilon \in \mathcal{D}$, if $\Upsilon(x) = \Upsilon(y)$, then $xI_\Upsilon y$.

Binary independence of irrelevant alternatives: For all $x, y \in X$ and for all $\Upsilon, \bar{\Upsilon} \in \mathcal{D}$, if $\Upsilon(x) = \bar{\Upsilon}(x)$ and $\Upsilon(y) = \bar{\Upsilon}(y)$, then

$$xR_\Upsilon y \Leftrightarrow xR_{\bar{\Upsilon}} y.$$

Anonymity: For all $\Upsilon, \bar{\Upsilon} \in \mathcal{D}$, if there exists a bijection $\rho\colon \{1, \ldots, n\} \to \{1 \ldots, n\}$ such that $\Upsilon_i = \bar{\Upsilon}_{\rho(i)}$ for all $i \in \{1, \ldots, n\}$, then $R_\Upsilon = R_{\bar{\Upsilon}}$.

Theorem 1: *Suppose F satisfies unlimited domain. F satisfies conditional Pareto indifference, binary independence of irrelevant alternatives and anonymity if and only if there* *exists an anonymous social-evaluation ordering R on Ω^n such that, for all $x, y \in X$ and for all $\Upsilon \in \mathcal{D}$,*

$$xR_\Upsilon y \Leftrightarrow \big(U(x), S(x), L(x)\big) R \big(U(y), S(y), L(y)\big).$$

The asymmetric and symmetric factors of the social-evaluation ordering R are denoted by P and I.

4. Intertemporal axioms and orderings

Intertemporal continuity: For all $(u, s, \ell) \in \Omega^n$, the sets $\{v \in \mathcal{R}^n \mid (v, s, \ell)R(u, s, \ell)\}$ and $\{v \in \mathcal{R}^n \mid (u, s, \ell)R(v, s, \ell)\}$ are closed in $\mathcal{R}^n$.

Intertemporal strong Pareto: For all $(u, s, \ell), (v, r, k) \in \Omega^n$,

(i) if $u = v$, then $(u, s, \ell)I(v, r, k)$;

(ii) if $u > v$, then $(u, s, \ell)P(v, r, k)$.

Conditional strong Pareto: For all $(u, s, \ell), (v, r, k) \in \Omega^n$,
(i) if $s = r$, $\ell = k$ and $u = v$, then $(u, s, \ell)I(v, r, k)$;
(ii) if $s = r$, $\ell = k$ and $u > v$, then $(u, s, \ell)P(v, r, k)$.

Birth-date conditional strong Pareto: For all $(u, s, \ell), (v, r, k) \in \Omega^n$,
(i) if $s = r$ and $u = v$, then $(u, s, \ell)I(v, r, k)$;
(ii) if $s = r$ and $u > v$, then $(u, s, \ell)P(v, r, k)$.

Lifetime conditional strong Pareto: For all $(u, s, \ell), (v, r, k) \in \Omega^n$,
(i) if $\ell = k$ and $u = v$, then $(u, s, \ell)I(v, r, k)$;
(ii) if $\ell = k$ and $u > v$, then $(u, s, \ell)P(v, r, k)$.

We require more notation to proceed. Let $i \in \{1, \ldots, n\}$, $(u, s, \ell) \in \Omega^n$ and $(u'_i, s'_i, \ell'_i) \in \Omega$. The vectors $v = (u_{-i}, u'_i) \in \mathcal{R}^n$, $r = (s_{-i}, s'_i) \in \mathcal{Z}^n_+$ and $k = (\ell_{-i}, \ell'_i) \in \{1, \ldots, \bar{L}\}^n$ are defined by

$$v_j = \begin{cases} u_j & \text{if } j \in \{1, \ldots, n\} \setminus \{i\}; \\ u'_j & \text{if } j = i, \end{cases}$$

$$r_j = \begin{cases} s_j & \text{if } j \in \{1, \ldots, n\} \setminus \{i\}; \\ s'_j & \text{if } j = i \end{cases}$$

and

$$k_j = \begin{cases} \ell_j & \text{if } j \in \{1, \ldots, n\} \setminus \{i\}; \\ \ell'_j & \text{if } j = i. \end{cases}$$

Individual intertemporal equivalence: There exists $\lambda_0 \in \{1, \ldots, \bar{L}\}$ such that, for all $(d, \sigma, \lambda) \in \Omega$ and for all $\sigma_0 \in \mathcal{Z}_+$, there exists $\hat{d} \in \mathcal{R}$

such that, for all $(u, s, \ell) \in \Omega^n$ and for all $i \in \{1, \ldots, n\}$,

$$\Big((u_{-i}, \hat{d}), (s_{-i}, \sigma_0), (\ell_{-i}, \lambda_0)\Big)\, I \, \big((u_{-i}, d), (s_{-i}, \sigma), (\ell_{-i}, \lambda)\big).$$

Birth-date conditional individual intertemporal equivalence: There exists $\lambda_0 \in \{1, \ldots, \bar{L}\}$ such that, for all $(d, \sigma) \in \mathcal{R} \times \mathcal{Z}_+$ and for all $\sigma_0 \in \mathcal{Z}_+$, there exists $\hat{d} \in \mathcal{R}$ such that, for all $(u, s) \in \mathcal{R}^n \times \mathcal{Z}_+^n$ and for all $i \in \{1, \ldots, n\}$,

$$\Big((u_{-i}, \hat{d}), (s_{-i}, \sigma_0), \lambda_0 \mathbf{1}_n\Big)\, I \big((u_{-i}, d), (s_{-i}, \sigma), \lambda_0 \mathbf{1}_n\big).$$

Lifetime conditional individual intertemporal equivalence: There exist $\sigma_0 \in \mathcal{Z}_+$ and $\lambda_0 \in \{1, \ldots, \bar{L}\}$ such that, for all $(d, \lambda) \in$

$\mathcal{R} \times \{1, \ldots, \bar{L}\}$, there exists $\hat{d} \in \mathcal{R}$ such that, for all $(u, \ell) \in \mathcal{R}^n \times \{1, \ldots, \bar{L}\}^n$ and for all $i \in \{1, \ldots, n\}$,

$$\Big((u_{-i}, \hat{d}), \sigma_0 \mathbf{1}_n, (\ell_{-i}, \lambda_0)\Big)\, I\big((u_{-i}, d), \sigma_0 \mathbf{1}_n, (\ell_{-i}, \lambda)\big).$$

R is a birth-date and lifetime dependent generalized-utilitarian ordering if and only if there exist a function $h\colon \Omega \to \mathcal{R}$, continuous and increasing in its first argument, and $\lambda_0 \in \{1, \ldots, \bar{L}\}$ such that $h(\mathcal{R}, \sigma_0, \lambda_0) \cap h(\mathcal{R}, \sigma, \lambda) \neq \emptyset$ for all $\sigma_0, \sigma \in \mathcal{Z}_+$ and for all $\lambda \in \{1, \ldots, \bar{L}\}$ and, for all $(u, s, \ell), (v, r, k) \in \Omega^n$,

$$(u, s, \ell) R (v, r, k) \Leftrightarrow \sum_{i=1}^{n} h(u_i, s_i, \ell_i) \geq \sum_{i=1}^{n} h(v_i, r_i, k_i).$$

Analogously, R is a birth-date dependent generalized-utilitarian ordering if and only if there exists a function $f\colon \mathcal{R} \times \mathcal{Z}_+ \to \mathcal{R}$, continuous and increasing in its first argument, such that $f(\mathcal{R}, \sigma_0) \cap f(\mathcal{R}, \sigma) \neq \emptyset$ for all $\sigma_0, \sigma \in \mathcal{Z}_+$ and, for all $(u, s, \ell), (v, r, k) \in \Omega^n$,

$$(u, s, \ell) R (v, r, k) \Leftrightarrow \sum_{i=1}^{n} f(u_i, s_i) \geq \sum_{i=1}^{n} f(v_i, r_i). \tag{4.1}$$

R is a lifetime dependent generalized-utilitarian ordering if and only if there exist a function $e: \mathcal{R} \times \{1, \ldots, \bar{L}\} \rightarrow \mathcal{R}$, continuous and increasing in its first argument, and $\lambda_0 \in \{1, \ldots, \bar{L}\}$ such that $e(\mathcal{R}, \lambda_0) \cap e(\mathcal{R}, \lambda) \neq \emptyset$ for all $\lambda \in \{1, \ldots, \bar{L}\}$ and, for all $(u, s, \ell), (v, r, k) \in \Omega^n$,

$$(u, s, \ell) R (v, r, k) \Leftrightarrow \sum_{i=1}^{n} e(u_i, \ell_i) \geq \sum_{i=1}^{n} e(v_i, k_i).$$

Finally, R is an intertemporal generalized-utilitarian ordering if and only if there exists a continuous and increasing function $g\colon \mathcal{R} \to \mathcal{R}$ such that, for all $(u, s, \ell), (v, r, k) \in \Omega^n$,

$$(u, s, \ell) R (v, r, k) \Leftrightarrow \sum_{i=1}^{n} g(u_i) \geq \sum_{i=1}^{n} g(v_i).$$

Independence of the utilities of the dead: For all $m \in \{1, \ldots, n-1\}$, for all $(u, s, \ell), (v, r, k) \in \Omega^m$, for all $(\bar{u}, \bar{s}, \bar{\ell}), (\bar{v}, \bar{r}, \bar{k}) \in \Omega^{n-m}$ and for all $t \in \mathcal{Z}_{++}$, if $\bar{s}_i + \bar{\ell}_i < t$ and $\bar{r}_i + \bar{k}_i < t$ for all $i \in \{1, \ldots, n-m\}$ and $s_i + 1 \geq t$ and $r_i + 1 \geq t$ for all $i \in \{1, \ldots, m\}$, then

$$\begin{aligned} &\left((u, \bar{u}), (s, \bar{s}), (\ell, \bar{\ell})\right) R \left((v, \bar{u}), (r, \bar{s}), (k, \bar{\ell})\right) \\ &\Leftrightarrow \left((u, \bar{v}), (s, \bar{r}), (\ell, \bar{k})\right) R \left((v, \bar{v}), (r, \bar{r}), (k, \bar{k})\right). \end{aligned}$$

Theorem 3:

(i) *An anonymous ordering R satisfies intertemporal continuity, conditional strong Pareto, individual intertemporal equivalence and independence of the utilities of the dead if and only if R is birth-date and lifetime dependent generalized-utilitarian.*

(ii) *An anonymous ordering R satisfies intertemporal continuity, birth-date conditional strong Pareto, birth-date conditional individual intertemporal equivalence and independence of the utilities of the dead if and only if R is birth-date dependent generalized-utilitarian.*

(iii) *An anonymous ordering R satisfies intertemporal continuity, lifetime conditional* *strong Pareto, lifetime conditional individual intertemporal equivalence and independence of the utilities of the dead if and only if R is lifetime dependent generalized-utilitarian.*

(iv) *An anonymous ordering R satisfies intertemporal continuity, intertemporal strong Pareto and independence of the utilities of the dead if and only if R is intertemporal generalized-utilitarian.*

Proof. We provide a detailed proof of Part (i). That the birth-date and lifetime dependent generalized-utilitarian orderings satisfy intertemporal continuity, conditional strong Pareto and independence of the utilities of the dead is straightforward to verify. The existence of a $\lambda_0 \in \{1, \ldots, \bar{L}\}$ such that $h(\mathcal{R}, \sigma_0, \lambda_0) \cap h(\mathcal{R}, \sigma, \lambda)$ is non-empty for all $\sigma_0, \sigma \in \mathcal{Z}_+$ and for all $\lambda \in \{1, \ldots, \bar{L}\}$, assumed in the definition of the orderings, guarantees that individual intertemporal equivalence is satisfied.

Now suppose R is an anonymous ordering satisfying the axioms of Part (i) of the theorem statement. The proof that R is birth-date and lifetime dependent generalized-utilitarian proceeds as follows. We define an ordering $\overset{*}{R}$ on $\mathcal{R}^n$ (that is, an ordering of utility vectors) as the restriction of R that is obtained by fixing birth dates and lengths of life at specific values. We then show that $\overset{*}{R}$ satisfies the axioms of Theorem 2 and, thus, must be generalized-utilitarian. Finally, we show that all comparisons according to R can be carried out by applying $\overset{*}{R}$ to utilities that depend on birth dates and lifetimes, resulting in birth-date and lifetime dependent generalized utilitarianism.

Let λ_0 be as in the definition of individual intertemporal equivalence. Define the ordering $\overset{*}{R}$ on $\mathcal{R}^n$ by letting, for all $u, v \in \mathcal{R}^n$,

$$u\overset{*}{R}v \Leftrightarrow (u, 0\mathbf{1}_n, \lambda_0\mathbf{1}_n)\, R\, (v, 0\mathbf{1}_n, \lambda_0\mathbf{1}_n)\,.$$

Clearly, $\overset{*}{R}$ is an anonymous ordering satisfying continuity and strong Pareto. The last remaining property of $\overset{*}{R}$ to be established is independence of the utilities of the unconcerned. Let $m \in \{1, \ldots, n-1\}$, $u, v \in \mathcal{R}^m$ and $\bar{u}, \bar{v} \in \mathcal{R}^{n-m}$. By repeated application of individual intertemporal equivalence, there exist $\hat{u}, \hat{v} \in \mathcal{R}^m$ such that

$$\big((\hat{u}, \bar{u}), (\bar{L}\mathbf{1}_m, 0\mathbf{1}_{n-m}), \lambda_0\mathbf{1}_n\big)\, I\, ((u, \bar{u}), 0\mathbf{1}_n, \lambda_0\mathbf{1}_n)\,, \tag{4.2}$$

$$\big((\hat{v}, \bar{u}), (\bar{L}\mathbf{1}_m, 0\mathbf{1}_{n-m}), \lambda_0\mathbf{1}_n\big)\, I\, ((v, \bar{u}), 0\mathbf{1}_n, \lambda_0\mathbf{1}_n)\,, \tag{4.3}$$

$$\big((\hat{u}, \bar{v}), (\bar{L}\mathbf{1}_m, 0\mathbf{1}_{n-m}), \lambda_0\mathbf{1}_n\big)\, I\, ((u, \bar{v}), 0\mathbf{1}_n, \lambda_0\mathbf{1}_n) \tag{4.4}$$

and

$$\big((\hat{v}, \bar{v}), (\bar{L}\mathbf{1}_m, 0\mathbf{1}_{n-m}), \lambda_0\mathbf{1}_n\big)\, I\, ((v, \bar{v}), 0\mathbf{1}_n, \lambda_0\mathbf{1}_n)\,. \tag{4.5}$$

(4.2) and (4.3) together imply

$$\begin{aligned} &((u, \bar{u}), 0\mathbf{1}_n, \lambda_0\mathbf{1}_n)\, R\, ((v, \bar{u}), 0\mathbf{1}_n, \lambda_0\mathbf{1}_n) \\ \Leftrightarrow\ &\big((\hat{u}, \bar{u}), (\bar{L}\mathbf{1}_m, 0\mathbf{1}_{n-m}), \lambda_0\mathbf{1}_n\big)\, R \\ &\big((\hat{v}, \bar{u}), (\bar{L}\mathbf{1}_m, 0\mathbf{1}_{n-m}), \lambda_0\mathbf{1}_n\big)\,. \end{aligned} \tag{4.6}$$

By independence of the utilities of the dead,

$$\begin{aligned}&\big((\hat{u},\bar{u}),(\bar{L}\mathbf{1}_m,0\mathbf{1}_{n-m}),\lambda_0\mathbf{1}_n\big)\,R\\&\big((\hat{v},\bar{u}),(\bar{L}\mathbf{1}_m,0\mathbf{1}_{n-m}),\lambda_0\mathbf{1}_n\big)\\\Leftrightarrow&\big((\hat{u},\bar{v}),(\bar{L}\mathbf{1}_m,0\mathbf{1}_{n-m}),\lambda_0\mathbf{1}_n\big)\,R\\&\big((\hat{v},\bar{v}),(\bar{L}\mathbf{1}_m,0\mathbf{1}_{n-m}),\lambda_0\mathbf{1}_n\big)\,.\end{aligned}\tag{4.7}$$

(4.4) and (4.5) together imply

$$\begin{aligned}&\big((\hat{u},\bar{v}),(\bar{L}\mathbf{1}_m,0\mathbf{1}_{n-m}),\lambda_0\mathbf{1}_n\big)\,R\\&\big((\hat{v},\bar{v}),(\bar{L}\mathbf{1}_m,0\mathbf{1}_{n-m}),\lambda_0\mathbf{1}_n\big)\\\Leftrightarrow&((u,\bar{v}),0\mathbf{1}_n,\lambda_0\mathbf{1}_n)\,R\\&((v,\bar{v}),0\mathbf{1}_n,\lambda_0\mathbf{1}_n)\,.\end{aligned}\tag{4.8}$$

Combining (4.6), (4.7) and (4.8), we obtain

$$\begin{aligned}&((u,\bar{u}),0\mathbf{1}_n,\lambda_0\mathbf{1}_n)\,R\,((v,\bar{u}),0\mathbf{1}_n,\lambda_0\mathbf{1}_n)\Leftrightarrow\\&((u,\bar{v}),0\mathbf{1}_n,\lambda_0\mathbf{1}_n)\,R\,((v,\bar{v}),0\mathbf{1}_n,\lambda_0\mathbf{1}_n)\,.\end{aligned}$$

By definition of $\overset{*}{R}$, this is equivalent to

$$(u, \bar{u})\overset{*}{R}(v, \bar{u}) \Leftrightarrow (u, \bar{v})\overset{*}{R}(v, \bar{v})$$

which establishes that independence of the utilities of the unconcerned is satisfied.

By Theorem 2, $\overset{*}{R}$ is generalized-utilitarian and, thus, there exists a continuous and increasing function $g: \mathcal{R} \to \mathcal{R}$ such that

$$u\overset{*}{R}v \Leftrightarrow \sum_{i=1}^{n} g(u_i) \geq \sum_{i=1}^{m} g(v_i)$$

for all $u, v \in \mathcal{R}^n$. Define the function $\bar{h}: \Omega \to \mathcal{R}$ by

$$\bar{h}(d, \sigma, \lambda) = \gamma \Leftrightarrow (d, \sigma, \lambda)I(\gamma, 0, \lambda_0)$$

for all $(d, \sigma, \lambda) \in \Omega$ and for all $\gamma \in \mathcal{R}$. This function is well-defined because R satisfies individual intertemporal equivalence.

Consider any $(u, s, \ell), (v, r, k) \in \Omega^n$. By repeated application of individual intertemporal equivalence,

$$\Big(\big(\bar{h}(u_i, s_i, \ell_i)\big)_{i=1}^{n}, 0\mathbf{1}_n, \lambda_0 \mathbf{1}_n \Big) I(u, s, l)$$

and

$$\Big(\big(\bar{h}(v_i, r_i, k_i)\big)_{i=1}^{n}, 0\mathbf{1}_n, \lambda_0 \mathbf{1}_n \Big) I(v, r, k).$$

Therefore,

$$
\begin{aligned}
&(u, s, \ell) R (v, r, k) \Leftrightarrow \\
&\Big(\left(\bar{h}(u_i, s_i, \ell_i)\right)_{i=1}^{n}, 0\mathbf{1}_n, \lambda_0 \mathbf{1}_n \Big) R \\
&\Big(\left(\bar{h}(v_i, r_i, k_i)\right)_{i=1}^{n}, 0\mathbf{1}_n, \lambda_0 \mathbf{1}_n \Big) \\
&\Leftrightarrow \\
&\left(\bar{h}(u_i, s_i, \ell_i)\right)_{i=1}^{n} \overset{*}{R} \left(\bar{h}(v_i, r_i, k_i)\right)_{i=1}^{n} \\
&\Leftrightarrow \\
&\sum_{i=1}^{n} g\left(\bar{h}(u_i, s_i, \ell_i)\right) \geq \sum_{i=1}^{n} g\left(\bar{h}(v_i, r_i, k_i)\right).
\end{aligned}
$$

Letting $h = g \circ \bar{h}$ (where $\circ$ denotes function composition), it follows that

$$
(u, s, \ell) R (v, r, k) \Leftrightarrow \sum_{i=1}^{n} h(u_i, s_i, \ell_i) \geq \sum_{i=1}^{n} h(v_i, r_i, k_i).
$$

That h satisfies $h(\mathcal{R}, \sigma_0, \lambda_0) \cap h(\mathcal{R}, \sigma, \lambda) \neq \emptyset$ for all $\sigma_0, \sigma \in \mathcal{Z}_+$ and for all $\lambda \in \{1, \ldots, \bar{L}\}$ follows from the definitions of $\bar{h}$ and h. This completes the proof of Part (i).

5. Geometric and linear discounting

The ordering R is geometric birth-date dependent generalized-utilitarian if and only if there exist a continuous and increasing function $g\colon \mathcal{R} \to \mathcal{R}$ and a constant $\delta \in \mathcal{R}_{++}$ such that, for all $(u, s, \ell), (v, k, r) \in \Omega^n$,

$$(u, s, \ell) R (v, r, k) \Leftrightarrow \sum_{i=1}^{n} \delta^{s_i} g(u_i) \geq \sum_{i=1}^{n} \delta^{r_i} g(v_i).$$

An alternative class of birth-date dependent orderings uses information on birth dates in a linear fashion. R is linear birth-date dependent generalized-utilitarian if and only if there exist a continuous and increasing function $g: \mathcal{R} \to \mathcal{R}$ and a constant $\beta \in \{-1, 1\}$ such that, for all $(u, s, \ell), (v, k, r) \in \Omega^n$,

$$(u, s, \ell)R(v, r, k)$$
$$\Leftrightarrow \sum_{i=1}^{n} g(u_i) + \beta \sum_{i=1}^{n} s_i \geq \sum_{i=1}^{n} g(v_i) + \beta \sum_{i=1}^{n} r_i.$$

In the case of $\beta = 1$, ceteris paribus, later births are considered better. If $\beta = -1$, the earlier people are born, the better the corresponding alternative is (provided that lifetime utilities are the same).

Stationarity: For all $(u, s, \ell), (v, k, r) \in \Omega^n$ and for all $\tau \in \mathcal{Z}_+$,

$$(u, s+\tau\mathbf{1}_n, \ell)R(v, r+\tau\mathbf{1}_n, k) \Leftrightarrow (u, s, \ell)R(v, r, k).$$

Theorem 4: *An anonymous ordering R satisfies intertemporal continuity, birth-date conditional strong Pareto, birth-date conditional individual intertemporal equivalence, independence of the existence of the dead and stationarity if and only if R is geometric or linear birth-date dependent generalized-utilitarian.*

6. Remarks

An argument that is sometimes made in favor of discounting is that very large sacrifices by those presently alive may be justified by larger gains to people who will exist in the distant future only. If these sacrifices are considered too demanding, discounting might be proposed to alleviate the negative effects on the generations that live earlier. However, this argument rests on the false claim that discounting necessarily increases the well-being of the present generation. To see that the claim is not true, consider a three-person society and suppose two alternatives x and y are such that person i is born in period i for all $i \in \{1, 2, 3\}$. In x, utility levels are $u_1 = 28$, $u_2 = 4$ and $u_3 = 44$ and, in y, lifetime utilities are $u_1 = u_2 = u_3 = 24$. If intertemporal generalized utilitarianism with the identity mapping as the transformation is used to evaluate the alternatives, x is better than y and the utility level of person 1, who represents the present generation, is 28. Alternatively suppose that geometric birth-date dependent generalized utilitarianism with the identity mapping and a discount factor of $\delta = 1/2$ is used instead. In that case, the sums of discounted utilities are $28 + 2 + 11 = 41$ for x and $24 + 12 + 6 = 42$

for y, so y is better and person 1's utility is 24.