Fundamentals of Mobility Measurement

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Outline

- Background
  - Basics
- Methods
  - General considerations
  - Status
  - Measures: Intuitive and others
- Principles
- Analysis
  - Basic structure
  - Consistency
  - Classes of measures
  - Data issues
  - Mobility indices: two classes
  - Decomposition
- Discussion and summary

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- a policy tool?
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- improving data on intra- and inter-generational mobility
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- What is known about mobility?

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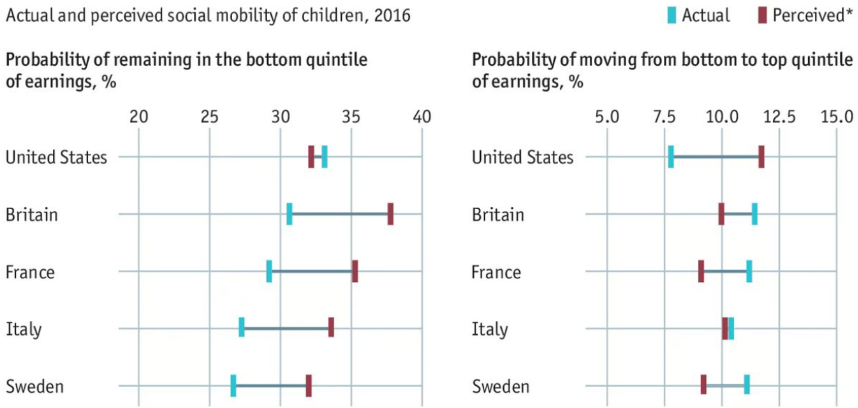
What do people know about mobility?

- do they value mobility?
- do they know it when they see it?

Perceptions and reality



Source: Alesina et al. (2018)

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- Variety of temporal context:
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  2. long term / volatility
- Variety of analytical context:
  - in relation to a specific dynamic model
  - in relation to social-welfare issues
  - as an abstract distributional concept

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- Appropriate tools?
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- An abstract distributional concept
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- This presentation
  - develops ideas in Cowell and Flachaire (2017, 2018)
  - shows how to give meaning of mobility comparisons

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Ingredients for a theory of mobility measurement:

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Ingredients for a theory of mobility measurement:

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Ingredient 1:

- Assume discrete time
- Focus on two periods: now (0) and the future (1)

Steps

- Separate the ingredients of problem
  1. time frame (two periods)
  2. status
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- Characterise an ordering
  - formulate principles as axioms
  - develop characterisation results

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First step in an approach to "status":

- define a finite set of K classes
- $n_k \geq 0$: # in class k, $k = 1, 2, \ldots, K$
- exclusive and exhaustive
- $\sum_{k=1}^{K} n_k = n$, the size of the population

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$k_0(i)$, $k_1(i)$: class occupied by person i at times t^0 and t^1

- mobility given by $\left(x_{k_0(1)}, ..., x_{k_0(n)}\right)$ and $\left(x_{k_1(1)}, ..., x_{k_1(n)}\right)$

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- cardinal attribute: just aggregate the xs?
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- Revaluing the income classes: $N_1(x_k) := \sum_{h=1}^{k} n_{1h},\ k = 1, ..., K$

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- *dynamic.* $z_i = \left(N_0\left(x_{k_0(i)}\right), N_1\left(x_{k_1(i)}\right)\right)$

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Exchange and structural mobility: (Van Kerm 2004, Tsui 2009)

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Statistical measures

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1. elasticity coefficient
   - linear regression of status-1 on status-0
   - $x_{1i} = \alpha + \beta x_{0i} + \varepsilon_i$
   - $1 - \hat{\beta}$ as a measure of mobility?

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2. correlation coefficient
   - use Pearson correlation coefficient $\hat{\rho}$
   - $1 - \hat{\rho}$ as a measure of mobility?

Statistical measures: elasticity coefficient

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Statistical measures: elasticity coefficient

- A high value of $1-\beta$ evidence of significant mobility?
- Low value does not necessarily imply low mobility
  - can have $1-\hat{\beta}=0$ where there is indeed mobility
  - since $\hat{\beta}=\frac{cov(\mathbf{x}_0,\mathbf{x}_1)}{var(\mathbf{x}_0)}$: $1-\hat{\beta}=0 \quad \Leftrightarrow \quad cov(\mathbf{x}_0,\mathbf{x}_1)=var(\mathbf{x}_0)$.

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- A difficulty:
  - take $\mathbf{x}_0=(x_{01},x_{01}+k,x_{01}+2k)$, $\mathbf{x}_1=(x_{11},x_{12},x_{11}+2k)$
  - we have $1-\hat{\beta}=0$, $\forall x_{01},x_{11},x_{12}$
- Example:
  - $\mathbf{x}_0=(1,2,3)$
  - $\mathbf{x}_1 \in \{(2,0,4),(2,1,4),(2,1760,4),(2100,1,2102),\dots\}$
  - zero mobility *in all cases*?

Statistical measures: correlation coefficient

- Both scale and translation independent:
  - if $x_1 = ax_0 + b$, then $\hat{\rho} = 1 \Leftrightarrow 1 - \hat{\rho} = 0$
  - so $\mathbf{x}_0 = (1, 2, 3)$ and $\mathbf{x}_1 = (0, 2, 4)$ imply $x_1 = 2x_0 - 2$; $1 - \hat{\rho} = 0$
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- Measure can behave strangely:
  - take equidistant status
  - $\mathbf{x}_0 = (x_{01}, x_{01} + k, x_{01} + 2k)$, $\mathbf{x}_1 = (x_{11}, x_{12}, x_{11})$
  - Get $1 - \hat{\rho} = 1$ and $1 - \hat{\beta} = 1$, $\forall x_{01}, x_{11}, x_{12}$

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- Example
  - $\mathbf{x}_0 = (1,2,3)$
  - $\mathbf{x}_1 \in \{(3,2,3),(3,0,3),(3,100,3),(1,2,1),(10,1,10),(2,1,2),\dots\}$
  - in all cases $1 - \hat{\rho} = 1$ and $1 - \hat{\beta} = 1$

Inequality-based measures

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  - $FO_2 = \frac{1}{n}\sum_{i=1} \mid \log y_{1i} - \log y_{0i} \mid$
- Shorrocks (1978) measures related to inequality:
  - $S_I = 1 - \frac{I(y_0+y_1)}{\frac{\mu_{y_0}}{\mu_{y_0+y_1}} I(y_0) + \frac{\mu_{y_1}}{\mu_{y_0+y_1}} I(y_1)}$
  - where $I(.)$ is a predefined inequality measure

Directional measures

- Ray and Genicot (2023) upward mobility index (absolute) :
  - $RG_1 = -\frac{1}{\alpha}\log\left(\frac{\sum_{i=1}^{n} y_{1i}^{-\alpha}}{\sum_{i=1}^{n} y_{0i}^{-\alpha}}\right)$, $\alpha > 0$

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- Ray and Genicot (2023) upward mobility index (relative):
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- Bárcena and Cantó (2025) downward mobility index
  - $BC_D = \frac{1}{n}\sum_{i\in D}\left(\frac{y_{0i}-y_{1i}}{y_{0i}}\right)^{\alpha}, \qquad \alpha \geq 0$

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- Bárcena and Cantó 2018 upward mobility index
  - $BC_{\mathrm{U}} = \frac{1}{n}\sum_{i\in U}\left(\frac{y_{1i}-y_{0i}}{y_{0i}}\right)^{\alpha}, \qquad \alpha \geq 0$

Comparative performance: 3-person society

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	period	period						
	0	1^a	1^b	1^c	1^d	1^e	1^f	1^g
A	10	20	15	20	40	25	10	10
B	20	40	25	40	80	45	30	40
C	40	80	45	10	20	15	40	160

Comparative performance: 3-person society

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B	20	40	25	40	80	45	30	40
C	40	80	45	10	20	15	40	160
Elasticity	$1-\hat{\beta}$	**0**	0.208	1.500	1.500	1.368	**0**	-1.000
Pearson correlation	$1-\hat{\rho}$	**0**	0.001	1.500	1.500	1.465	0.053	**0**
Fields-Ok 1	FO_1	23.333	5.000	20.000	36.667	21.667	3.333	46.667
Fields-Ok 2	FO_2	0.693	0.249	0.924	1.155	0.903	0.135	0.693
Shorrocks 1	S_{Theil}	**0**	0.011	0.736	0.680	0.739	0.034	0.053
Shorrocks 2	S_{Gini}	**0**	**0**	0.500	0.444	0.500	**0**	**0**
Ray-Genicot absolute	RG_1	0.693	0.306	**0**	0.693	0.306	0.100	0.288
Ray-Genicot relative	RG_2	**0**	0.112	**0**	**0**	0.112	-0.033	-0.811
Bárcena-Cantó downward	BC_{D}	**0**	**0**	0.250	0.167	0.208	**0**	**0**
Bárcena-Cantó upward	BC_{U}	1.000	0.292	0.667	2.000	0.917	0.167	1.333

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	1989-2000	*2000-2011*
$1-\beta$	0.7564	0.6928
$1-\rho$	0.7947	0.7257
FO_1	6506.5	16979.62
FO_2	0.9619	1.1726

Principles: movement

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  2. inequality change and exchange mobility (Jäntti and Jenkins 2015; Kessler and Greenberg 1981, McClendon 1977)
- Essential for mobility measurement?
  - ensures a minimum-mobility property
  - situation with some movement registers higher mobility than a situation without movement

Principles: decomposition

- Applied to other aspects of distributional analysis
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- Several aspects of decomposability may be desirable
  - decomposition by population characteristics
  - decomposition by region
- Special for mobility:
  - decompose by direction
  - mobility in terms of upward and downward movements (Bárcena and Cantó 2018; Bárcena and Cantó 2025)

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  - comparing one bivariate distribution of (status-in-0, status-in-1) with another

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- Under such circumstances should each pair of distributions be ranked the same?

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- General considerations
- Status
- Measures: Intuitive and others

Principles

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Discussion and summary

Axiomatic approach

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- Basic concepts
  - status
    - individual observation
    - derived from distribution
  - Individual i's status history $z_i = (u_i, v_i)$
  - profile: a list of histories $\mathbf{z} = (z_1, z_2, ...z_n)$

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- Key axioms:
  - correspond to main principles
  - movement, decomposition consistency
  - do this in two stages

Monotonicity

[Monotonicity] Let $\mathbf{z}, \mathbf{z}' \in Z^n$ differ only in their ith history and $u_i' = u_i$ and define two conditions $\mathscr{U} :=$ "$v_i > v_i' \geq u_i$" and $\mathscr{D} :=$ "$v_i < v_i' \leq u_i$". If z_i satisfies either $\mathscr{U}$ or $\mathscr{D}$ then $\mathbf{z} \succ \mathbf{z}'$

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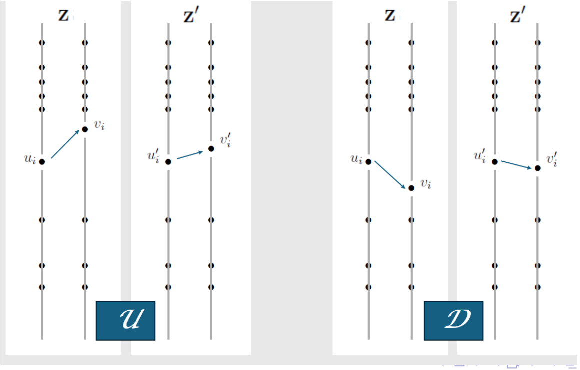

Independence

[Independence] Consider two profiles $\mathbf{z}, \mathbf{z}' \in Z^n$ where there is some $i \in \{2, \ldots, n-1\}$ such that $u_{i-1} < u_i < u_{i+1}$, $v_{i-1} < v_i < v_{i+1}$, $u'_{i-1} < u_i < u'_{i+1}$, $v'_{i-1} < v_i < v'_{i+1}$. Let $\mathbf{z}(\zeta, i)$ denote the profile formed by replacing the ith history in $\mathbf{z}$ by the history $\zeta \in Z$ and let $\hat{Z}_i := [u_{i-1}, u_{i+1}] \times [v_{i-1}, v_{i+1}]$. If $\mathbf{z} \sim \mathbf{z}'$ and $z_i = z'_i$ then $\mathbf{z}(\zeta, i) \sim \mathbf{z}'(\zeta, i)$ for all $\zeta \in \hat{Z}_i \bigcap \hat{Z}'_i$.

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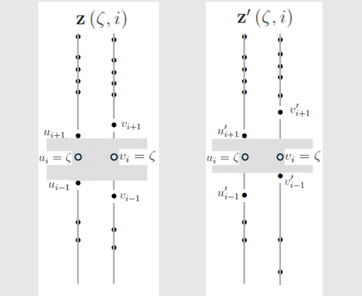

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- ***[Continuity]*** $\succeq$ is continuous on Z^n
- ***[Local immobility]*** Let $\mathbf{z}, \mathbf{z}' \in Z^n$ where for some i, $u_i = v_i$, $v_i' = u_i'$ and, for all $j \neq i$, $u_j' = u_j$, $v_j' = v_j$. Then $\mathbf{z} \sim \mathbf{z}'$

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Theorem 1: given these axioms then $\forall \mathbf{z} \in Z^n$ the mobility ordering $\succeq$ is an increasing monotonic transform of $\sum_{i=1}^{n} \phi_i(z_i)$

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Implication of SI (1)

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- **Theorem 2:** $\succeq$ is representable by $\sum_{i=1}^{n} \phi_i(z_i)$, where the evaluation function ϕ_i is given by

$$\phi_i(u,v) = \begin{cases} A_i(v)\left[u^\alpha - v^\alpha\right] & \text{if OSI holds,} \\ A_i'(u)\left[v^\alpha - u^\alpha\right] & \text{if DSI holds,} \\ v^\beta h_i\left(\frac{u}{v}\right) & \text{if PSI holds,} \end{cases}$$

where A_i, A_i', h are functions of one variable and α, β are constants.

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- $h_i\left(\frac{u}{v}\right) = A_i(v)\left[u^{\alpha}v^{-\beta} - v^{\alpha-\beta}\right]$

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- Provides the basis for a class of mobility measures

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where A_i, A_i', g_i, g_i' are functions of one variable, a is a constant, and $g_i(0) = 0$

- Combining PTI with OTI or DTI :

$$\text{either } \phi_i(u,v) = a_i[v-u],$$

$$\text{or } \phi_i(u,v) = a_i\left[e^{\beta[v-u]} - 1\right]$$

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- Then $\phi_i (u_i, v_i)$ must take the form

$$\phi_i (u, v) = a_i [v - u],$$

where a_i is a constant, specific to each history

- The term a_i can do a lot of work!

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- If representation of $\succeq$ is constant under replication: $c(n) = c'/n$
- Normalise: $\beta = 1$, $c' = \frac{1}{\alpha[\alpha-1]}$, impose mobility property:
- Mobility is given by $\psi\left(\frac{1}{n}\sum_{i=1}^n u_i^\alpha v_i^{1-\alpha} - \theta(\mu_u,\mu_v), \mu_u, \mu_v\right)$
  - $\mu_u := \frac{1}{n}\sum_{i=1}^n u_i$, $\mu_v := \frac{1}{n}\sum_{i=1}^n v_i$, $\theta(\mu,\mu) = \mu$

Implications: Class 2

- Evaluation function: $\phi_i(u,v) = a_i[v-u]$

Implications: Class 2

- Evaluation function: $\phi_i(u,v) = a_i[v-u]$
  - for any history $z_i = (v_i, u_i)$
  - write the (signed) distance $d_i := v_i - u_i$
- Overall mobility index $\sum_{i=1}^{n} a_i d_{(i)}$
  - $d_{(i)}$ is the ith component of vector $(d_1, ..., d_n)$ in ascending order
  - $d_{(1)} < 0$ is greatest downward mobility
  - $d_{(n)} > 0$ is greatest upward mobility
- Monotonicity?
  - $a_i < 0$ whenever $d_{(i)} < 0$
  - $a_i > 0$ whenever $d_{(i)} > 0$
- Population
  - a_i should be proportional to $1/n$
  - up to a change in scale we have $\frac{1}{n}\sum_{i=1}^{n} a_i d_{(i)}$

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- Measures so far consistent with Interpretation 1 of "movement"
- Now consider how to handle Interpretation 2 of "movement"

Mean-normalised data (1)

[Monotonicity-2] *If* $\mathbf{z}, \mathbf{z}'$ *differ only in components i, j:* $u_i' = u_i$, $u_j' = u_j$, $v_i' - v_i = v_j - v_j'$; *then, if* $v_i > v_i' \geq u_i$ *and if* $v_j < v_j' \leq u_j$, $\mathbf{z} \succ \mathbf{z}'$

Mean-normalised data (1)

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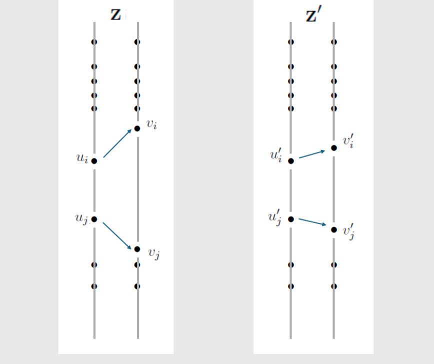

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- Mean-normalised version
  - divide each u_i by μ_u and each v_i by μ_v

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- Parameter gives sensitivity to types of mobility.
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- Concerned with *ranks* not *income levels*? Make status ordinal:
    - use estimated distribution function

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- In limit, as c as $c\to\infty$: $M'_\beta := \frac{1}{n\beta^2}\sum_{i=1}^{n}\left[e^{\beta[u_i-\mu_u-v_i+\mu_v]}-1\right]$.
  - remains unchanged with abolute additions to status
  - β is a sensitivity parameter

M_α as a function of α: example

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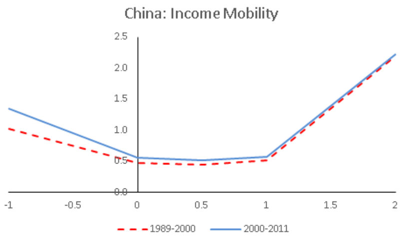

Mobility measures: Class 2

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  - $a_i = \begin{cases} -1 & \text{if } i < i^* \\ +1 & \text{if } i \geq i^* \end{cases}$ where i^* is largest i s.t. $d_{(i)} < 0$
  - then measure becomes $\Gamma_0 := \frac{1}{n} \sum_{i=1}^{n} \left| d_{(i)} \right|$
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- **Case (b)**
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- **Special Case (b): linear ϕ**
  - weights are: $a_i = \frac{i}{n} - p - \frac{1}{2n}$
  - so $\Gamma_1 := \frac{1}{n} \sum_{i=1}^{n} \frac{i}{n} d_{(i)} - \left[p + \frac{1}{2n}\right] \mu_d$,
  - $\Gamma_1 = {}^1\!/\!_2 G + \mu_d \left[\frac{1}{2} - p\right]$

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Decomposition: issues

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  - get different types of decomposition
  - depends on interpretation of the independence axiom
- Aspects in common with decomposition in other fields (social welfare, inequality):
  - by personal characteristics
  - by regions or countries
- Special for mobility:
  - Up and Down
  - similar to rich / poor decomposition for inequality
  - often treated as defining aspects of mobility
  - U: Ray and Genicot (2023); D: Bárcena and Cantó (2025)

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  - $p_k \left[\frac{\mu_{u,k}}{\mu_u}\right]^\alpha \left[\frac{\mu_{v,k}}{\mu_v}\right]^{1-\alpha}$ weight on group k
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- Partition population into "upward" U and "downward" D groups:
  - $M_\alpha = w^{\mathsf{U}} M_\alpha^{\mathsf{U}} + w^{\mathsf{D}} M_\alpha^{\mathsf{D}} + M_\alpha^{\mathsf{btw}}$
  - compare Bárcena and Cantó (2025), Ray and Genicot (2023)eq (27)

Decomposition: Class 1 (Example)

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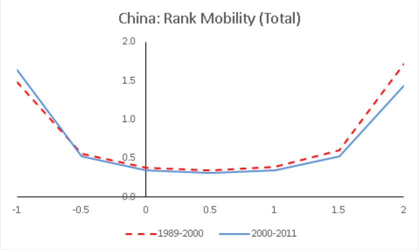

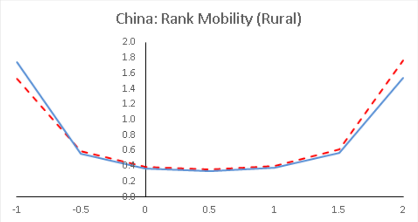

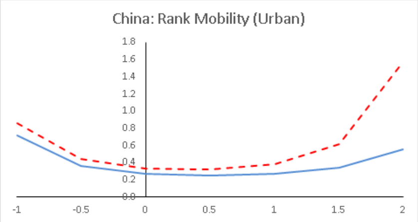

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- FO indices:
  - $\Gamma_0 = -pd^{\mathsf{D}} + [1-p]\, d^{\mathsf{U}}$
- For general case $a_i = \phi\left(\frac{i}{n} - p - \frac{1}{2n}\right)$
  - $\Gamma_\gamma = p^{\gamma+1}\Gamma_\gamma^{\mathsf{D}} + [1-p]^{\gamma+1}\Gamma_\gamma^{\mathsf{U}}$
  - $\phi(x) = x^\gamma$

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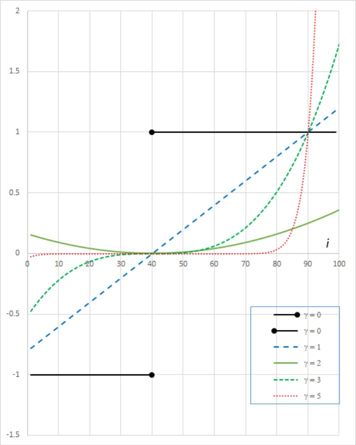

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- Blackorby and Donaldson (1980, Bossert and Pfingsten (1990, Cowell and Flachaire (2021)

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  - focus on meaning of mobility comparisons.
  - characterise “suitable” measures
- The results
  - two broad classes of mobility indices
  - each class satisfies the minimal set of requirements for mobility comparisons
  - each of these classes has a natural interpretation in terms of distributional analysis

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