The measurement of inequality and household equivalence scales

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1. Introduction

inequality

social welfare

homogeneous-heterogeneous population

living standard

equivalent income

equivalence scales

Example (Glewwe 1991)

1 person $X_1 = 28$

6 persons (1 adult + 5 children) $X_2 = 72$

1 person $X_3 = 400$

weights $w_1 = 1, \quad w_2 = 6, \quad w_3 = 1$

equivalence scales $m_1 = 1, \quad m_2 = 2, \quad m_3 = 1$

equivalent incomes X_i / m_i

distribution $(28, 36, 36, 36, 36, 36, 36, 400)$

redistribution $X_1 \to 28 + 2, \quad X_2 \to 72 - 2$

new distribution $(30, 35, 35, 35, 35, 35, 35, 400)$

Theil measure $\uparrow\uparrow$

Present paper

1. Introduction
2. Homogeneous population
3. Heterogeneous population
4. Transformation: Artificial population
5. Some arguments for using scales
6. Equivalence scales
7. Isoelastic scales
8. Conclusion

2. Homogeneous population

n **(identical) individuals,** $i = 1, \ldots, n$

income $X_i \in D$**, where** $D = \mathbb{R}$ **or** $D = \mathbb{R}_{++}$

income distribution $\boldsymbol{X} = \left(X_1, \ldots, X_n \right)$

social welfare function $W\left(\boldsymbol{X} \right) = \Sigma_i U\left(X_i \right)$

Definition: progressive transfer

Axiom PT: progressive transfer

Theorem: W **satisfies** $PT \Leftrightarrow U\left(X \right)$ **strictly concave**

$$\mathcal{W} := \left\{ W\left(\boldsymbol{X} \right) = \Sigma_i U\left(X_i \right) \quad \middle| U \text{ strictly concave} \right\}$$

HOM A: **For all** $\boldsymbol{X}, \boldsymbol{Y} \in D^n$ **s.t.** $\mu(\boldsymbol{X}) = \mu(\boldsymbol{Y})$ **the following statements are equivalent**

(1) $\boldsymbol{Y} \underset{PT}{\longleftarrow} \ldots \underset{PT}{\longleftarrow} \boldsymbol{X}^1 \underset{PT}{\longleftarrow} \boldsymbol{X}$

(2) **There is a bistochastic matrix:** $\boldsymbol{Y} = B \cdot \boldsymbol{X}$

(3) **Lorenz-dominance:** $\boldsymbol{Y} >_{LD} \boldsymbol{X}$

(4) $W(\boldsymbol{Y}) > W(\boldsymbol{X})$ **for all** $W \in \mathcal{W}$

HOM B: **For all** $\boldsymbol{X}, \boldsymbol{Y} \in D^n$ **the following statements are equivalent**

(5) **Generalized Lorenz-dominance:** $\boldsymbol{Y} >_{GLD} \boldsymbol{X}$

(6) $W(\boldsymbol{Y}) > W(\boldsymbol{X})$ **for all** $W \in \mathcal{W}$

(Relative) inequality measures

measures $I(\boldsymbol{X})$

ethical measures $I(\boldsymbol{X}) = \left(\mu(\boldsymbol{X}) - \xi(\boldsymbol{X})\right) \big/ \mu(\boldsymbol{X})$

Examples: Atkinson-class

Generalized entropy class

etc.

Lorenz-consistency

References: Atkinson (JET 1970), Dasgupta/Sen/Starrett (JET 1973), Marshall/Olkin (1979), Shorrocks (Economica 1983)

3. Heterogeneous population

Living standard

K **types,** $i = 1, \ldots, K$

type i**:** $a_i \in \mathbb{R}^M$

$A = \{a_1, \ldots, a_K\}$

Household/individual i**:** (X_i, a_i)

Comparison by means of $\succsim_{LS}$ **(complete ordering)**

(L1) continuous

(L2) strictly increasing in income

(L3) comparability

(L4) Existence of differentiable representations

$\mathcal{L} := \left\{ \succsim_{LS} \middle| \succsim_{LS} \text{ fulfills (L1)-(L4)} \right\}$

$L(X,a)$ **representation**

equivalent income function

$E^r(X,a) := L^{-1}\left(L(X,a), a_r\right)$

$r =$ **reference type**

(E0) $E^r(X, a_r) = X$

(E1) continuous

(E2) strictly increasing in income

(E3) comparability

Examples

$$E^r\left(X,a\right)=X\big/m\left(a\right)$$

$$E^r\left(X,a\right)=X-b(a)$$

$$E^r\left(X,a\right)=\left(1/\delta\right)\ln\left(1+\alpha\left(a_r\right)/\alpha\left(a\right)\left(e^{\delta X}-1\right)\right)\ \textbf{for}\ \delta>0$$

Generation of orderings

set of utility functions

microeconomic foundation

subjective choice

econometric estimation

institutional norms

income distribution $\left(\boldsymbol{X},\boldsymbol{a}\right)=\left(X_1,\ldots,X_K,a_1,\ldots,a_K\right)$

social welfare function $W\left(\boldsymbol{X},\boldsymbol{a}\right)=\Sigma_i V\left(X_i,a_i\right)$

Definition: progressive transfer between types

$\left(\boldsymbol{Y},\boldsymbol{a}\right)\leftarrow\left(BTPT\right)\left(\boldsymbol{X},\boldsymbol{a}\right)$ **if** $Y_i=X_i+\varepsilon,\ Y_j=X_j-\varepsilon,\ X_k=Y_k$

for $k\neq i,j$ **and** $E^r\left(X_i,a_i\right)<E^r\left(Y_i,a_i\right)\leq E^r\left(Y_j,a_j\right)<E^r\left(X_j,a_j\right)$

Axiom $BTPT\left(\succsim_{LS}\right)$**:** **progressive transfer between types**

Theorem: For all $\succsim_{LS} \in \mathcal{L}$**.**

W **satisfies** $BTPT\left(\succsim_{LS}\right) \Leftrightarrow V\left(X,a\right)$ **strictly concave and** $-V'\left(X,a\right)$ **represents** $\succsim_{LS}$**.**

Examples

$V\left(X,a\right) = \alpha\left(a\right)X^{\varepsilon}/\varepsilon$ **for** $\varepsilon < 1$

$V\left(X,a\right) = -\alpha\left(a\right)e^{-\gamma X}$ **for** $\gamma > 0$

$V\left(X,a\right) = \alpha\left(a\right)\ln\left(1-e^{-\delta X}\right)$ **for** $\delta > 0$

References: Coulter/Cowell/Jenkins (BER 1992), Ebert (ITAX 2000), Ebert (JPET 2008)

4. Transformation: artificial population

$$(X,a) \to \left(E^r(X,a), w(a)\right)$$

$E^r(X,a)$ **equivalent income**

$w(a)$ **(absolute) weight**

Examples: $w(a)=1$

$w(a)=n(a)$ **number of individuals**

$w(a)=m(a)$ **equivalence scale**

Glewwe's example

5. Some arguments for using equivalence scales

Ebert (MASS 1995)

needs ranking

nested Atkinson welfare function $WA(\boldsymbol{X})$

generalized Pigou-Dalton principle GPD **(upwards/downwards)**

aggregation principle AGG

Proposition

$WA(\boldsymbol{X})$ **satisfies** GPD **and** AGG

$\Leftrightarrow$ **“equivalence scales and weights = scales”**

Ebert/Moyes (Ectra 2003)

needs ranking

path-independence

reference-independence

Proposition

(Generalized) Lorenz dominance is reference-independent ⇔ "equivalence scales"

Proposition

(Generalized) Lorenz dominance satisfies BTPT

⇔ "equivalence scales and weights = scales"

Ebert/Moyes (JET 2000)

needs ranking

path-independence

tax-system

reference-independence

overall inequality reduction

Proposition

Let the tax system be overall inequality reducing.

Taxation is reference-independent

⇔ “equivalence scales”

Ebert (SCW 2004)

needs ranking

rank-dependent social welfare ordering

property welfare *WELF*

principle of population *PP*

scale invariance *REL*

axiom *BTPT*

Proposition

Welfare ordering satisfies *WELF*, *PP*, *REL*, *BTPT*

$\Leftrightarrow$ **"equivalence scales and weights = scales"**

6. Relative equivalence scales: artificial homogenisation

n **(heterogeneous) individuals/households,** $i = 1, \dots, n$

individual/household i**:** $\left(X_i, m\left(\overline{a}_i\right)\right)$ **with** $\overline{a}_i \in A$

transformation of the income distribution

$$\left(X_1, \dots, X_n, \quad \overline{a}_1, \dots \overline{a}_n\right) \Rightarrow \left(X_1 / m_1, \cdots, X_n / m_n, \quad m_1, \cdots, m_n\right)$$

where $m_i := m\left(\overline{a}_i\right)$ **equivalence scale**

X_i / m_i **equivalent income**

$w\left(\overline{a}_i\right) = m_i$ **absolute weight**

$\boldsymbol{m} = (m_1, \ldots, m_n)$

income distribution $(\boldsymbol{X}, \boldsymbol{m})$

social welfare function $W(\boldsymbol{X}, \boldsymbol{m}) = \Sigma_i V(X_i, m_i)$

Definition: $BTPT(\boldsymbol{m})$

Axiom $BTPT(\boldsymbol{m})$

Theorem: For all $\boldsymbol{m}$.

W **satisfies** $BTPT(\boldsymbol{m}) \Leftrightarrow V(X, m) = mU(X/m)$ **and** U **strictly concave.**

$$\mathcal{W}(\boldsymbol{m}) = \left\{ W(\boldsymbol{X}, \boldsymbol{m}) = \Sigma_i m_i U(X_i/m_i) \quad \middle| U \text{ strictly concave} \right\}$$

Example

$$W(\boldsymbol{X}, \boldsymbol{a}) \quad = \Sigma_i \alpha(a) X_i^{\varepsilon} / \varepsilon \quad = \Sigma_i m_i (X_i/m_i)^{\varepsilon} / \varepsilon \quad = W(\boldsymbol{X}, \boldsymbol{m})$$

with $\quad m_i = \alpha_i^{1/(1-\varepsilon)} \quad$ **and** $\quad U(X) = X^{\varepsilon}/\varepsilon$

Generalization

Definition: progressive redistribution scheme $\boldsymbol{Y} \leftarrow \boldsymbol{X}$

s.t.

$$Y_i / m_i = \Sigma_j \lambda_{ij} \left(X_j / m_j \right), \quad \Sigma_j \lambda_{ij} = 1, \text{ and } \quad \Sigma_i Y_i = \Sigma_i X_i$$

Axiom $PRS(\boldsymbol{m})$

Theorem: For all $\boldsymbol{m}$**.**

W **satisfies** $PRS(\boldsymbol{m}) \Leftrightarrow V(X, m) = mU(X/m)$ **and** U **strictly concave.**

Suppose that $X_i / m_i \le X_{i+1} / m_{i+1}$ **for** $i = 1, \ldots, n-1$

Definition: Lorenz curve

$$L(\boldsymbol{X}, i) = \sum_{j=1}^{i} \frac{m_j}{\Sigma m_k} \left(X_j / m_j \right) \Big/ \mu(\boldsymbol{X}, \boldsymbol{m})$$

Definition $B = \left(b_{ij} \right)$ **is** $\boldsymbol{m}$**-stochastic**

if

$$\sum_i b_{ij} = 1 \qquad \text{for } j = 1, \ldots, n$$

$$\sum_j b_{ij} m_j = m_i \quad \text{for } i = 1, \ldots, n$$

HET A: **For all** $\boldsymbol{m}$ **and** $\boldsymbol{X}, \boldsymbol{Y} \in D^n$ **with** $\mu(\boldsymbol{X}, \boldsymbol{m}) = \mu(\boldsymbol{Y}, \boldsymbol{m})$ **the following statements are equivalent**

(1) $\boldsymbol{Y} \xleftarrow[PRS(\boldsymbol{m})]{} \boldsymbol{X}$

(2) There exists an $\boldsymbol{m}$**-stochastic matrix:** $\boldsymbol{Y} = B \cdot \boldsymbol{X}$

(3) Lorenz-dominance: $(\boldsymbol{Y}, \boldsymbol{m}) >_{LD} (\boldsymbol{X}, \boldsymbol{m})$

(4) $W(\boldsymbol{Y}, \boldsymbol{m}) > W(\boldsymbol{X}, \boldsymbol{m})$ **for all** $W \in \mathcal{W}(\boldsymbol{m})$

HET B: **For all** $\boldsymbol{m}$ **and all** $\boldsymbol{X}, \boldsymbol{Y} \in D^n$ **the following statements are equivalent**

(5) Generalized Lorenz-dominance: $(\boldsymbol{Y}, \boldsymbol{m}) >_{GLD} (\boldsymbol{X}, \boldsymbol{m})$

(6) $W(\boldsymbol{Y}, \boldsymbol{m}) > W(\boldsymbol{X}, \boldsymbol{m})$ **for all** $W \in \mathcal{W}(\boldsymbol{m})$

Inequality measures

heterogeneous population

weighted income distribution $\left(X_1/m_1,\ldots,X_n/m_n, \quad m_1,\ldots m_n \right)$

$\mu\left(\boldsymbol{X},\boldsymbol{m}\right)$ **weighted arithmetic mean**

Example: Atkinson-measure

$$I\left(\boldsymbol{X},\boldsymbol{m}\right)=1-\frac{\left(\Sigma_i \frac{m_i}{\Sigma m_k}\left(\frac{X_i}{m_i}\right)^{\varepsilon}\right)^{1/\varepsilon}}{\Sigma_i \frac{m_i}{\Sigma m_k}\left(\frac{X_i}{m_i}\right)}$$

References: Ebert (SCW 1999), Ebert (SCW 2004), Ebert (JOEI 2007)

7. Isoelastic scales (with Patrick Moyes)

n **individuals (adults)**

$m(n) = n^{\delta}$ **for** $0 \leq \delta \leq 1$

$\delta = 0$ **no difference**

$\delta = 1$ **“individualism”**

$0 < \delta < 1$ **private/public goods**

References: Buhmann/Rainwater/Schmaus/Smeeding (RIW 1988), Coulter/Cowell/Jenkins (EJ 1992), Ebert/Moyes (JPOP 2009)

Characterization

$$I\left(\boldsymbol{X},\boldsymbol{n}\right)=J\left(E\left(\boldsymbol{X},\boldsymbol{n}\right),w\left(\boldsymbol{n}\right)\right)$$

“equivalenc scales and weight = scale”

Normalization N $J\left(\boldsymbol{s},\boldsymbol{w}\right)=0 \Leftrightarrow s_1=\ldots=s_n$

scale invariance SI $J\left(\lambda\boldsymbol{s},\boldsymbol{w}\right)=J\left(\boldsymbol{s},\boldsymbol{w}\right)$

distribution invariance DI $J\left(\boldsymbol{s},\lambda\boldsymbol{w}\right)=J\left(\boldsymbol{s},\boldsymbol{w}\right)$

neediness scale invariance $I\left(\boldsymbol{X},\boldsymbol{n}\right)=I\left(\boldsymbol{X},\kappa\boldsymbol{n}\right)$

Proposition:

Consider *I* where *J* satisfies N, SI and DI.

***I* satisfies neediness scale invariance**

$$\Leftrightarrow m\left(n\right)=\beta n^{\delta}$$

Derivation of equivalence scales

household with *n* individuals

x **private good**

y **public good**

prices p**,** q**; income** M

$U\left(x,G\right)$ **utility function**

$G = y/\psi\left(\Theta,n\right)$ **effective consumption**

Θ **= degree of publicness**

$\psi\left(\Theta,n\right)$ **congestion function**

The household maximizes

$$U\left(x,G\right) \text{ s.t. } pnx+\psi\left(\Theta,n\right)qG=M$$

expenditure function $E\left(p,q,n,\Theta,u\right)$

equivalence scale

$$m\left(n\right)=\frac{E\left(p,q,n,\Theta,u\right)}{E\left(p,q,1,\Theta,u\right)}$$

Proposition

$m\left(n\right)$ **is isoelastic**

$\Leftrightarrow \psi\left(\Theta,n\right)=n^{\delta\left(\Theta\right)}$ **and** U **Cobb-Douglas**

8. Conclusion

homogenisation

unidimensional/multidimensional criteria

complete/incomplete orderings

inequality concept

welfare, inequality, poverty