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## **The redistribution of income when needs differ**

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## 1. Introduction

The paper deals with some criteria which allow us to compare income distributions when households can differ in income and needs. We confine ourselves to investigating the redistribution of income among individuals or households belonging to the same population. Here at least two approaches can be thought of. First, one possibility is to examine particular forms of redistribution and to describe and evaluate them. For instance one can discuss transfers or income tax schedules. Then we compare the income distribution generated by the transfer/tax schedule with the original income distribution. Second, another possibility is to introduce and to investigate (more) general criteria which are appropriate for a comparison of arbitrary income distributions. In this case one can use an indicator, like e.g. a social welfare function or an inequality measure, or a dominance criterion like Lorenz dominance. In both approaches the criteria employed<sup>1</sup> can always be treated as normative or as positive concept.

The topic of redistribution is well settled for a *homogeneous* population in which the individuals may differ only with respect to income. In this part of welfare economics progressive transfers form the basic ingredients of most investigations. They are defined by a rank-preserving transfer of a (small) amount of income from a richer to a poorer individual. This kind of redistribution can be described positively by a corresponding matrix. Normatively, one can e.g. derive the properties of a social welfare function being improved by this type of transfer. The relationship between a progressive transfer and Lorenz dominance or dominance with respect to certain classes of welfare functions can be examined. One can also admit additional increases in income. Furthermore, as a slight generalization, one can consider the redistribution of income by means of income tax schedules. These problems have been dealt with among others by Atkinson (1970), Dasgupta, Sen, and Starrett (1973), Jakobsson (1976), and Shorrocks (1983).

It took some time to realize that things are much more complicated if there is some *heterogeneity* and if the individuals or households considered may additionally possess different attributes implying different needs. Glewwe (1991) was one of the first authors demonstrating that in such a (multidimensional) framework progressive transfers can increase inequality (and decrease welfare), i.e., they do not necessarily improve, but can deteriorate the situation. The fundamental problem with a heterogeneous population of (differing) households is the fact that income is no longer a reliable indicator of living standard. The households' needs have also to be taken into account. Here one can choose again between two alternative

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<sup>1</sup> Lambert (2001) presents an excellent survey of the literature on (the problems related to) the redistribution of income.

approaches. In the first one the heterogeneous income distribution is transformed into an artificial one-dimensional distribution and then the usual criteria known for a homogeneous population can be employed. In practice equivalence scales – or more generally equivalent income functions – are often applied. They allow us to derive a distribution of equivalent income which is a representation of the living standard attained. Then this distribution of equivalent income forms the basis of further investigations. In the second approach living standards are not derived explicitly, but they are taken into account implicitly by the criteria employed. For instance, Atkinson and Bourguignon (1987) suggest sequential generalized Lorenz dominance. It is based on a comparison of heterogeneous income distributions by means of a class of welfare functions which reflect the differences in the households' attributes.<sup>2</sup> Below we will follow the first approach: Living standard will be defined explicitly by an additional ordering.

The objective of this paper is to provide a brief survey of the basic results on the redistribution of income when households differ in needs. Section 2 collects the fundamental results available for a homogeneous population. This presentation serves as a background later on. In section 3 the problems associated with the heterogeneity of a population are discussed. In particular the meaning and definition of living standard are explored in a general framework. Section 4 deals with the evaluation of redistribution in a heterogeneous population. In this section the analysis is confined to a particular setting. Living standard is measured by means of equivalence scales which are most popular in applied work. Finally section 5 discusses some extensions and concludes.

## **2. Redistribution in a homogeneous population**

At first we examine a homogeneous population consisting of individuals who have the same attributes and the same needs. We assume that the population is fixed and that there are  $n \geq 2$  individuals. Their numbering is not relevant. We define  $N := \{1, \dots, n\}$ .  $X_i \in D$  denotes individual  $i$ 's income where the set of feasible income  $D$  is equal to  $\mathbb{R}$  or  $\mathbb{R}_{++}$ . Negative incomes are also admitted in order to make the results in this and the following sections comparable. Let  $\mathbf{X} = (X_1, \dots, X_n) \in D^n$  be an income distribution.

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<sup>2</sup> Bourguignon (1989) employs a similar approach. Ebert (2010a) examines the connections between both approaches.

## 2.1 Living standard, social welfare and redistribution

In a homogeneous population the comparison of living standards is simple. Since the individuals considered do not differ with respect to their attributes we can identify an individual's living standard with her income; i.e., individual  $j$  is (weakly) better off than individual  $i$  if and only if  $X_i \leq X_j$ . Later on we will see that a comparison is more difficult for a heterogeneous population.

As far as social welfare is concerned we confine ourselves to a well-known class of welfare functions  $W$ . We assume that (total) social welfare can be represented by

$$W(\mathbf{X}) = \sum_{i=1}^n U(X_i) \quad (1)$$

where  $U : D \rightarrow \mathbb{R}$  is a utility-of-income function which is assumed to be continuous and strictly increasing. As there are no differences between the individuals' attributes the welfare function is symmetric and anonymous in incomes. It is, furthermore, additively separable. This assumption is made for simplicity. Many results on welfare functions presented below can be extended to (the class of) nonseparable welfare functions.

Now we investigate the redistribution of income within this framework. We employ progressive transfers, the classical form of the redistribution of income and we define

### Definition 1

$\mathbf{Y} \in D^n$  is obtained from  $\mathbf{X} \in D^n$  by a progressive transfer if there are  $\varepsilon > 0$  and  $i, j \in N$ ,  $i \neq j$  such that  $Y_i = X_i + \varepsilon$ ,  $Y_j = X_j - \varepsilon$ , and  $Y_h = X_h$  for  $h \in N - \{i, j\}$  and  $X_i < Y_i < Y_j < X_j$ .

Here a small amount of income is redistributed from a richer individual to a poorer one. Their ranking in terms of income or living standard is not changed. But the difference in living standard between  $i$  and individual  $j$  is reduced. For completeness it is mentioned that a (progressive) transfer does not change total income.

Then the question arises how such a transfer is to be evaluated. It is answered by the principle of progressive transfers:

### Principle of progressive transfers PT

Assume that  $\mathbf{Y} \in D^n$  is generated from  $\mathbf{X} \in D^n$  by a progressive transfer. Then  $W(\mathbf{Y}) > W(\mathbf{X})$ .

This principle requires that social welfare increases. It goes back to Pigou (1912) and Dalton (1920) and is therefore sometimes called the Pigou-Dalton principle of transfers.

Having introduced the concepts of living standard, social welfare function and progressive transfer we are now able to present a first result:

**Theorem 1**

*A social welfare function  $W(\mathbf{X}) = \sum U(X_i)$  satisfies the principle PT if and only if the utility-of-income function  $U(X)$  is strictly concave.*

Thus we obtain a restriction on the social welfare function. If the principle of progressive transfers is to be satisfied the utility-of-income function  $U$  and therefore the corresponding social welfare function  $W(X)$  has to be strictly concave. The utility gain implied by a given amount of additional income decreases with the initial income. There are several interpretations for this utility and welfare function in the literature (cf. e.g. Lambert (2001)). Welfare functions having this property are often called inequality averse. For later use we introduce the set

$$\mathcal{W} := \{W(\mathbf{X}) = \sum U(X_i) \mid U \text{ strictly concave}\}. \tag{2}$$

It contains all inequality averse (additively separable) social welfare functions.

**2.2 Characterizations**

In the next step we want to examine the meaning of progressive transfers in more detail. Our analysis is partially descriptive and partially normative. At first we describe the redistribution of income mathematically. Here the notion of a bistochastic matrix is required.

**Definition 2**

*A matrix  $B = (b_{ij}) \in \mathbb{R}_{++}^{n \times n}$  is called bistochastic if and only if  $\sum_i b_{ij} = 1$  and  $\sum_j b_{ij} = 1$  for all  $j \in N$  and, respectively,  $i \in N$ .*

Obviously all column and row sums have to be equal to unity. Thus an income vector which has been generated by means of a bistochastic matrix  $B$  has some nice properties. Suppose that  $\mathbf{Y} = B \cdot \mathbf{X}$  for  $\mathbf{X}, \mathbf{Y} \in D^n$ . Then each income  $Y_i = \sum_j b_{ij} X_j$  (for  $i = 1, \dots, n$ ) is a convex sum of the original incomes  $X_j$ . Furthermore the total income is not changed, since

$$\sum_i Y_i = \sum_i \sum_j b_{ij} X_j = \sum_j \left( \sum_i b_{ij} \right) X_j = \sum_j X_j.$$

So the transformation corresponds to some redistribution of income. We then obtain<sup>3</sup>

### HOM 1

*Assume that  $X, Y \in D^n$ . Then  $Y$  is obtained from  $X$  by a (finite) sequence of progressive transfers if and only if there is a bistochastic matrix  $B \in \mathbb{R}_{++}^{n \times n}$ , which is not a permutation matrix, such that  $Y = B \cdot X$ .*

This result allows us to describe the repeated application of a progressive transfer to an income distribution precisely. This kind of redistribution is equivalent to the application of a bistochastic matrix to the corresponding income vector. But the converse statement might be more surprising. A sequence of progressive transfers corresponds to (the multiplication by) a bistochastic matrix.

Obviously a progressive transfer diminishes the inequality between the two individuals concerned. It is not yet clear whether overall inequality is also decreased. The inequality between the two incomes which are changed and some other incomes possibly increases. A possibility to describe the inequality inherent to an income distribution is using a Lorenz curve. This curve is formally defined by  $LC(i/n, X) := \sum_{j=1}^i X_{(j)} / \mu(X)$  for  $i = 0, \dots, n$  where  $X_{( )}$  is the income vector in which the incomes of  $X$  are rearranged such that  $X_{(i)} \leq X_{(i+1)}$  for  $i = 1, \dots, n-1$  and  $\mu(X)$  denotes the average income. The curve is interpolated linearly between  $i/n$  and  $(i+1)/n$  for  $i = 0, \dots, n-1$ . It is well known that the Lorenz curve is equal to a 45° degree line defined on the interval  $[0,1]$  if and only if all incomes are identical (i.e., if there is no inequality). Therefore we can describe an income distribution (and its inequality) graphically by its Lorenz curve. Moreover, the area between the Lorenz curve and the 45° line is equal to one half of the Gini coefficient which is a prominent measure of inequality. The Lorenz curve can, however, also be employed to define a quasi-ordering which – as we will show below – has normative significance:

### Definition 3

*$Y \in D^n$  (strictly) Lorenz dominates  $X \in D^n$ , i.e.,  $Y \geq_{LD} X$  ( $Y >_{LD} X$ ) if and only if  $LC(i/n, Y) \geq LC(i/n, X)$  for  $i = 0, \dots, n$  (and at least one inequality is strict).*

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<sup>3</sup> There are various sources in the literature for the derivation of HOM 1-HOM 4, cf. e.g. Atkinson (1970) and Dasgupta, Sen, and Starrett (1973). Compare also Marshall and Olkin (1979).

Lorenz dominance is, of course, an incomplete relation since Lorenz curves can intersect. The question raised can be answered since there is an important relationship between Lorenz dominance and the principle of progressive transfers. We have

## **HOM 2**

*Assume that  $X, Y \in D^n$  and  $\mu(X) = \mu(Y)$ . Then  $Y$  is obtained from  $X$  by a (finite) sequence of progressive transfers if and only if  $Y$  strictly Lorenz dominates  $X$ .*

Thus (repeated) application of a progressive transfer also implies a reduction in (overall) inequality. Conversely, if an income distribution  $Y$  Lorenz dominates  $X$ , the former can be recovered from  $X$  in a finite number of steps by redistributing income from a richer to a poorer individual. This second part demonstrates that Lorenz dominance is closely related to a favorable redistribution of income.

Up to now, the results HOM 1 and HOM 2 provide only further descriptions of progressive transfers. But there is also a normative link. One can establish

## **HOM 3**

*Assume that  $X, Y \in D^n$  and  $\mu(X) = \mu(Y)$ . Then  $Y$  is obtained from  $X$  by a (finite) sequence of progressive transfer if and only if  $W(Y) > W(X)$  for all  $W \in \mathcal{W}$ .*

The class  $\mathcal{W}$  contains the inequality averse social welfare functions. Therefore it is not surprising that a (sequence of) progressive transfer increases social welfare. On the other hand, the converse statement is not so obvious. If  $\mu(X) = \mu(Y)$  one can define a quasi-ordering by postulating  $W(Y) > W(X)$  for all  $W \in \mathcal{W}$ . Unanimity among all inequality averse welfare functions implies that the better income distribution can be constructed from the worse one by (a sequence of) progressive transfers. Here we get a normative justification of this kind of redistribution.

Combining the last two results we also obtain a normative characterization of Lorenz dominance:

## **HOM 4**

*Assume that  $X, Y \in D^n$  and  $\mu(X) = \mu(Y)$ . Then  $Y >_{LD} X$  if and only if  $W(Y) > W(X)$  for all  $W \in \mathcal{W}$ .*

The Lorenz curve can be interpreted as an appropriate tool for the comparison of (the inequality of) two income distributions. In this sense it is a positive concept. HOM 4 shows that this kind of comparison has also a normative content. Total and average income is not changed by any redistribution. For completeness we now consider income distributions which may have different means. Then we need a different concept of dominance. At first we define the generalized Lorenz curve by  $GLC(i/n, \mathbf{X}) := \sum_{j=1}^i X_{(j)}$  for  $i = 0, \dots, n$  and by linear interpolation between  $i/n$  and  $(i+1)/n$ . It allows us to introduce

#### **Definition 4**

$Y \in D^n$  (strictly) generalized Lorenz dominates  $X \in D^n$ , i.e.,  $Y \geq_{GL} X$  ( $Y >_{GL} X$ ), if and only if  $GLC(i/n, Y) \geq GLC(i/n, X)$  for  $i = 0, \dots, n$  (and at least one inequality is strict).

Generalized Lorenz curves may also intersect, but dominance can be characterized by

#### **HOM 5**

Assume that  $X, Y \in D^n$ . Then  $Y >_{GL} X$  if and only if  $W(Y) > W(X)$  for all  $W \in \mathcal{W}$ .

This result (cf. Shorrocks (1983)) examines the impact of the redistribution of income and of income changes (increases). It is interesting since it combines distributional considerations with efficiency aspects. Therefore the transition from Lorenz dominance to generalized Lorenz dominance allows one to consider distributions having different average incomes. Given that the social welfare functions are (a priori) strictly increasing in income HOM 5 provides a description of second degree stochastic dominance.

Finally we want to discuss the redistribution of income by means of income taxation. Let  $t: D \rightarrow \mathbb{R}$  be an income tax schedule which is differentiable (for convenience). Since we consider only identical households such a schedule is horizontally equal by definition, i.e., in our framework horizontal equity is satisfied from the beginning. But the conditions for progression have to be introduced. We have

#### **Definition 5**

A tax schedule  $t$  is progressive if and only if its residual elasticity  $R(X) := d(X - t(X))/dX \cdot X / (X - t(X))$  is less than or equal to unity for all  $X \in D$ .



Thus a tax schedule is progressive if the relative change in net income is less than or equal to the relative change in gross income on the relevant domain. It turns out that a progressive tax schedule is inequality reducing. We obtain<sup>4</sup>

### HOM 6

*A tax schedule  $t$  is progressive if and only if  $\mathbf{X} - t(\mathbf{X}) >_{LD} \mathbf{X}$  for all income distributions  $\mathbf{X} \in D^n$ .*

For a progressive tax the distribution of net income always Lorenz dominates the distribution of gross income, i.e., it possesses less inequality. This result demonstrates that the degree of progression of a tax schedule is closely related to the kind of redistribution implied by the tax. Exactly the progressive tax schedules are inequality reducing.

The discussion in this section clarifies the meaning of redistribution in a homogeneous population and relates various concepts. An interpretation in terms of social welfare can be given. The connection between progressive transfers and progressive income taxation is revealed. The meaning and the relevance of Lorenz and generalized Lorenz dominance are demonstrated.

## 3. Heterogeneous population

Next we extend the framework and deal with redistribution in a heterogeneous population.<sup>5</sup> We assume that at least some individuals have different attributes or that some households are different with respect to size, composition, and/or needs. Let there be  $K$  different types. In order to be definite and to concentrate on the heterogeneity we consider exactly one household of each type. This assumption will be dropped in the next section. I.e., here we assume that the population is fixed and consists of  $K$  households. Household  $k$  possesses type  $a_k \in \mathbb{R}^M$ . Let  $A$  denote the set of attributes  $\{a_1, \dots, a_K\}$ . The parameter  $a_k$  describes a household's attributes like the number of children, the number of adults, their needs etc.  $X_k \in D$  is now household  $k$ 's total income. Collecting this information we define  $(\mathbf{X}, \mathbf{a}) := (X_1, \dots, X_K, a_1, \dots, a_K)$  where  $\mathbf{X} \in \mathbb{R}^K$  and  $\mathbf{a} \in \mathbb{R}^{K \cdot M}$ .  $(\mathbf{X}, \mathbf{a})$  denotes a heterogeneous income distribution. Since in this section there is exactly one household of type  $k$ , it is not really necessary to mention the attributes  $\mathbf{a}$ , but the notation proves helpful later on.

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<sup>4</sup> This result goes back to Fellman (1976) (sufficiency), Jakobsson (1976), and Kakwani (1977).

<sup>5</sup> This topic is treated in more detail in Ebert (2008) which also contains a proof of Theorem 2.

As above we want to introduce a transfer and to consider its consequences in the extended framework. Since the households are different the relationship between household income and living standard is no longer obvious. Therefore we have to discuss the definition of living standard in more detail. A household  $k$  is described by  $(X_k, a_k)$ , its income and its attributes. We assume that all household members attain the same standard of living which depends on  $(X_k, a_k)$ . In order to determine this living standard and to be able to compare the living standard of different household types additional information is required. We therefore introduce an explicit ordering  $\succsim_{LS}$  defined on  $\mathcal{D} := D \times A$  which reflects the households' situation. We will say that household  $l$  with income  $X_l$  and type  $a_l$  is (weakly) better off than household  $k$  with income  $X_k$  and type  $a_k$  if and only if we have  $(X_k, a_k) \succsim_{LS} (X_l, a_l)$ . The relation  $\succsim_{LS}$  is assumed to be complete, reflexive, and transitive. Let  $\succ_{LS}$  denote the asymmetric and  $\sim_{LS}$  its symmetric part. This ordering has to satisfy three basic properties:

(L1)  $\succsim_{LS}$  is continuous.

(L2)  $\succsim_{LS}$  is strictly monotonic in income, i.e.,  $(\bar{X}_k, a_k) \succ_{LS} (X_k, a_k)$  whenever  $\bar{X}_k > X_k$  for  $a_k \in A$ .

(L3) For all  $(X_k, a_k) \in \mathcal{D}$  and  $k, l$  there is  $X_l \in D$  such that  $(X_k, a_k) \sim (X_l, a_l)$ .

These properties make sense. (L1) is a regularity condition. (L2) postulates that any increase in income increases the living standard. (L3) guarantees that every (feasible) living standard can be attained by each type. Furthermore, given (L1)-(L3) the ordering  $\succsim_{LS}$  can be represented equivalently by an indicator  $L: \mathcal{D} \rightarrow \mathbb{R}$  which is continuous, strictly increasing in income and satisfies a range condition, i.e., we have  $L(D, a_k) = L(D, a_l)$  for all  $k, l$ . In this section we confine ourselves to orderings  $\succsim_{LS}$  which satisfy (L4): They can be represented by once continuously differentiable indicators  $L$ . We then define the class of feasible orderings of living standard by  $\mathcal{L} := \{\succsim_{LS} \mid \succsim_{LS} \text{ satisfies (L1)-(L4)}\}$ .

There are infinitely many ways of defining an ordering  $\succsim_{LS}$  (cf. e.g. Coulter, Cowell, and Jenkins (1992), Ebert (2000a), and Ebert and Moyes (2009)). For the following discussion we always assume that one ordering is given. By definition a representation  $L$  is ordinal. In practice often a particular representation is chosen. It is defined by

$$E^r(X_k, a_k) := L^{-1}(L(X_k, a_k), a_r) \quad (3)$$

for  $a_k, a_r \in A$  and  $X_k \in D$ . Here  $L^{-1}$  is the inverse of  $L$  with respect to income. The type  $a_r$  can be interpreted as a reference type since  $E^r(X_k, a_k)$  is equal to the income a household of type  $a_r$  needs in order to be as well off as a household of type  $a_k$  with income  $X_k$ . This income is called household  $k$ 's equivalent income – given the reference type  $a_r$  – and  $E^r(X, a)$  is called an equivalent income function (cf. Ebert (2000a), Ebert and Moyes (2000, 2003), and Donaldson and Pendakur (2004)). Equivalent income is also a representation of the living standard attained.

For empirical work often equivalence scales are employed. Relative equivalence scales are represented by  $E^r(X, a) = X/m(a)$ , absolute equivalence scales by  $E^r(X, a) = X - b(a)$ . By definition we have  $m(a_r) = 1$  and  $b(a_r) = 0$  for the reference type. These scales are income independent, but depend on the choice of the reference type. Moreover the functions are linear in income. Of course, other nonlinear equivalent income functions (and underlying orderings of living standard) can be defined. Below we will consider  $E^r(X, a) = (1/\delta) \ln(1 + \alpha(a_r)/\alpha(a)(e^{\delta X} - 1))$  for  $\delta > 0$  where  $\alpha: \mathbb{R}^M \rightarrow \mathbb{R}_{++}$  takes into account the households' attributes. This equivalent income function is comparable to the use of relative scales for low and of absolute scales for high levels of income (cf. Ebert (2000a)).

Having discussed the problem of living standard in a heterogeneous population we now introduce a class of corresponding social welfare functions. We consider functions having the form

$$W(X, a) = \sum_{k=1}^K V(X_k, a_k) \quad (4)$$

where  $V: \mathcal{D} \rightarrow \mathbb{R}$  is a household utility (or utility-of-household-income) function which is assumed to be twice continuously differentiable and strictly increasing in income. A household's contribution to social welfare depends on its type. Therefore this welfare function is no longer symmetric in income. Differentiability means that the utility and social welfare functions are smooth. This assumption is made only in this section and is helpful in deriving the relationships below. The welfare function is also additively separable. This property in particular guarantees that a household's (marginal) utility does not depend on the remaining households' situation.

In a third step we assume that a reference type  $a_r$  is given and consider transfers between households. Here the problem arises that income is no longer a valid indicator of living standard. Therefore progressive transfers are based on equivalent income. We define

**Definition 6**

$(Y, \mathbf{a}) \in D^K \times \{\mathbf{a}\}$  is obtained from  $(X, \mathbf{a}) \in D^K \times \{\mathbf{a}\}$  by a between-type progressive transfer with respect to  $\succsim_{LS}$  if there are  $\varepsilon > 0$  and  $k, l, k \neq l$ , such that  $Y_k = X_k + \varepsilon$ ,  $Y_l = X_l - \varepsilon$ , and  $Y_h = X_h$  for  $h \neq k, l$  and

$$E^r(X_k, a_k) < E^r(Y_k, a_k) < E^r(Y_l, a_l) < E^r(X_l, a_l).$$

Thus a between-type progressive transfer is a transfer of a small amount of income from a better off household to a worse off household which leaves their ranking in terms of equivalent income unchanged. There are no restrictions on the types involved; only the equivalent incomes are decisive.

As above we suggest a corresponding principle of transfers:

**Principle of between-type progressive transfers BTPT( $\succsim_{LS}$ )**

Assume that  $(Y, \mathbf{a}) \in D^K \times \{\mathbf{a}\}$  is generated from  $(X, \mathbf{a}) \in D^K \times \{\mathbf{a}\}$  by a between-type progressive transfer w.r.t.  $\succsim_{LS}$ . Then  $W(Y, \mathbf{a}) > W(X, \mathbf{a})$ .

BTPT( $\succsim_{LS}$ ) requires that such a transfer increases welfare. This principle is a generalization of the conventional Pigou-Dalton principle of transfers to the extended framework. Repeated application of this principle demonstrates that a heterogeneous income distribution in which the living standard of all households is the same is an optimal distribution – given any welfare function satisfying BTPT( $\succsim_{LS}$ ).

Now we are able to examine the relationship between a welfare function and an ordering of living standard precisely in the heterogeneous framework. We obtain

**Theorem 2**

Let  $\succsim_{LS} \in \mathcal{L}$ . The social welfare function  $W(X, \mathbf{a}) = \Sigma V(X_k, a_k)$  satisfies BTPT( $\succsim_{LS}$ ) if and only if the utility-of-household-income function  $V(X, a)$  is strictly concave and  $-V'(X, a)$  represents the ordering  $\succsim_{LS}$  (where  $V'$  is the derivative of  $V(X, a)$  w.r.t. income).

Again strict concavity of the utility function is a necessary condition for the satisfaction of BTPT ( $\succsim_{LS}$ ). But this condition is no longer sufficient. There must be a definite connection between the definition of living standard and the social welfare function. The indicator  $-V'(X, a)$ , which is essentially the social marginal utility-of-household-income function, has to imply the same ordering as  $\succsim_{LS}$ ; i.e., this ordering can be recovered from the welfare function. It is well known that social marginal utility is closely related to the equity preference inherent in a welfare function. Thus for BTPT ( $\succsim_{LS}$ ) the equity preference and the definition of living standard have to be consistent with one another. We define the class  $\mathcal{W}(\succsim_{LS}) := \{W(\mathbf{X}, \mathbf{a}) = \sum V(X_k, a_k) \mid V(\mathbf{X}, \mathbf{a}) \text{ twice continuously differentiable, strictly concave, and strictly increasing and } -V'(\mathbf{X}, \mathbf{a}) \text{ represents } \succsim_{LS}\}$ . It is obvious that in the heterogeneous framework the class of inequality averse welfare functions can be decomposed into infinitely many subclasses, each depending on an ordering of living standard. By the way, it is obvious that Theorem 2 collapses to Theorem 1 if all household types are identical. Then there is only one ordering  $\succsim_{LS}$ .

At this place it may be illuminating to consider some examples. If we use an Atkinson-type function and set  $V(X, a) = \alpha(a)X^\varepsilon/\varepsilon$  where  $\alpha(a) > 0$  and  $\varepsilon < 1$ ,  $\varepsilon \neq 0$  we obtain relative equivalence scales, i.e.,  $E^r(X, a) = (\alpha(a)/\alpha(a_r))^{1/\varepsilon} X =: X/m(a)$ . For a Kolm-Pollak function ( $V(X, a) = -\alpha(a)e^{-\gamma X}$  for  $\gamma > 0$ ) we get absolute equivalence scales, i.e.,  $E^r(X, a) = X - ((1/\gamma)\ln(\alpha(a)/\alpha(a_r))) =: X - b(a)$ . When we choose  $V(X, a) = \alpha(a)\ln(1 - e^{-\delta X})$  for  $\delta > 0$  and  $\alpha(a) > 0$  we obtain the nonlinear equivalent income functions mentioned in the previous section. These examples also demonstrate that the (sub)classes  $\mathcal{W}(\succsim_{LS})$  may be relatively small.

Since the social marginal utility-of-household-income function defines exactly one ordering, a given social welfare function is compatible with exactly one ordering  $\succsim_{LS}$  of living standard. Conversely, if an ordering  $\succsim_{LS}$  is given, it is possible to derive the class of welfare functions satisfying BTPT ( $\succsim_{LS}$ ). Indeed, we only have to find those welfare functions whose social marginal utility represents  $\succsim_{LS}$  (cf. Ebert (2008) on these points).

## 4. Redistribution in a heterogeneous population: Using equivalence scales

In the previous section we have extended the analysis to a heterogeneous population having different household types. We have investigated the most general framework one can think of, since arbitrary orderings of living standard have been admitted. Furthermore, Theorem 2 characterizes the welfare functions satisfying the principle of between-type progressive transfers for any ordering  $\succ_{LS}$ . Given the background of section 2 one would like to derive further results which are analogous or similar to HOM 1-HOM 6. But here we face several difficulties. First, the concepts required, like e.g. the matrix describing the redistribution in income, have not yet been developed for the general model. Second, the tools used – e.g. Lorenz dominance – are well defined only for particular orderings of living standard – like those based on equivalence scales (cf. Ebert and Moyes (2003)). Third, income tax schedules must possess a specific form – which is based on equivalence scales – in order to be inequality reducing (cf. Ebert and Moyes (2000)). These formal and theoretical considerations suggest to confine the analysis to relative equivalence scales, or at least to particular orderings of living standard. Besides that most empirical work also uses this specific approach of measuring living standard in a heterogeneous population. Therefore in this section we will reconsider the problem of redistribution for the case that living standard is represented by equivalence scales. The approach of using equivalence scales is so attractive in theoretical and empirical work since it essentially allows to transform a (heterogeneous) multidimensional distribution into a (homogeneous) weighted one-dimensional distribution of equivalent income. We will then be able to perform an analysis of the redistribution of income which is analogous to the examination in section 2 for a homogeneous population.<sup>6</sup>

### 4.1 Living standard, social welfare and redistribution

We again assume that there are  $K$  household types described by the attributes  $a_1, \dots, a_K$  and that there is a reference type  $a_r$  (often identical with single adults). Furthermore, we suppose that the relative equivalence scales  $m(a_1), \dots, m(a_K)$  are given and that  $m(a_r) = 1$ . Let the population now consist of  $n$  households. Household  $i$ , for  $i = 1, \dots, n$ , is characterized by its income  $X_i$  and its type  $\bar{a}_i$ . For brevity the corresponding equivalence scale is denoted by  $m_i := m(\bar{a}_i)$ . Then the heterogeneous income distribution can be described by

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<sup>6</sup> The discussion in this section is based on Ebert (1999) which also contains the proofs of most results presented.

$(\mathbf{X}, \mathbf{m}) = (X_1, \dots, X_n, m_1, \dots, m_n)$ . It is in principle a multidimensional distribution. The scales  $m_i$  reflect the household's attributes and needs. The households may have arbitrary types, i.e., now it is possible that several households of the same type are considered.

In this framework living standard is characterized by equivalent income. Household  $j$  is better off than household  $i$  if and only if  $X_i/m_i \leq X_j/m_j$  for  $i, j \in N = \{1, \dots, n\}$ . Next we describe the class of welfare functions. We will use the social welfare functions

$$W(\mathbf{X}, \mathbf{m}) = \sum_{i=1}^n V(X_i, m_i)$$

where  $V$  is twice continuously differentiable and strictly increasing in income. Here  $V(\cdot, m_i)$  is an abbreviation of  $V(\cdot, \bar{a}_i)$  for  $i = 1, \dots, n$ .

Finally we can reformulate the between-type transfer and the corresponding principle for this particular case:

**Definition 7**

$(\mathbf{Y}, \mathbf{m}) \in D^n \times \{\mathbf{m}\}$  is obtained from  $(\mathbf{X}, \mathbf{m}) \in D^n \times \{\mathbf{m}\}$  by a between-type progressive transfer if there are  $\varepsilon > 0$  and  $i, j \in N$ ,  $i \neq j$  such that  $Y_i = X_i + \varepsilon$ ,  $Y_j = X_j - \varepsilon$  and  $Y_h = X_h$  for  $h \neq i, j$  and

$$X_i/m_i < Y_i/m_i < Y_j/m_j < X_j/m_j$$

and

**Principle of between-type progressive transfers BTPT(m)**

Assume that  $(\mathbf{Y}, \mathbf{m}) \in D^n \times \{\mathbf{m}\}$  is generated from  $(\mathbf{X}, \mathbf{m}) \in D^n \times \{\mathbf{m}\}$  by a between-type progressive transfer. Then  $W(\mathbf{Y}, \mathbf{m}) > W(\mathbf{X}, \mathbf{m})$ .

These definitions take into account the particular form of the underlying equivalent income function. Obviously a between-type progressive transfer is an appropriate generalization of a simple progressive transfer defined for a homogeneous population.

Then we also obtain an analogue to Theorem 1 and Theorem 2 which is proved in Ebert (1997a).

**Theorem 3**

Let  $m(a_1), \dots, m(a_k)$  be given. The social welfare function  $W(\mathbf{X}, \mathbf{m}) = \sum V(X_i, m_i)$  satisfies *BTPT*( $\mathbf{m}$ ) if and only if there is a strictly concave and increasing utility-of-income function  $U(X)$  such that  $V(X, m) = mU(X/m)$ .

As we have to expect,  $V$  has to be strictly concave. Furthermore the utility-of-household-income function has to possess a particular form which guarantees that the welfare function is compatible with the measurement of living standard by means of the relative equivalence scales given.

From Theorem 2 we know that the social marginal utility has to represent the living standard. We obtain

$$\begin{aligned} -V'(X_i, m_i) \leq -V'(X_j, m_j) &\Leftrightarrow -m_i U'(X_i/m_i)/m_i \leq -m_j U'(X_j/m_j)/m_j \\ &\Leftrightarrow U'(X_i/m_i) \geq U'(X_j/m_j) \Leftrightarrow X_i/m_i \leq X_j/m_j. \end{aligned}$$

Thus this condition is also satisfied.

The above discussion shows that Atkinson-type welfare functions belong to relative equivalence scales since it turns out that

$$\sum \alpha_i X_i^\varepsilon / \varepsilon = \sum m_i (X_i/m_i)^\varepsilon / \varepsilon \quad \text{if } m_i = \alpha_i^{\frac{1}{1-\varepsilon}},$$

i.e., this welfare function possesses the form characterized in Theorem 3. We have

$$W(\mathbf{X}, \mathbf{m}) = \sum_{i=1}^n m_i U(X_i/m_i) \tag{5}$$

where  $U$  is twice continuously differentiable, strictly concave and strictly increasing in income. If we choose single adults as reference type and interpret  $U$  as an individual utility-of-income function, then a single adult's utility depends on the household's living standard (equivalent income)  $X_i/m_i$ . Furthermore, this utility is weighted by  $m_i$ , i.e.,  $m_i$  can be interpreted as the number of equivalent adults in household  $i$ . More formally spoken, the heterogeneous income distribution  $(\mathbf{X}, \mathbf{m})$  is transformed into  $(\bar{\mathbf{X}}, \mathbf{m})$ , a distribution of equivalent income  $\bar{\mathbf{X}} := (X_1/m_1, \dots, X_n/m_n)$  where  $\mathbf{m} := (m_1, \dots, m_n)$  denotes the (absolute) weights. Household  $i$  corresponds to a household of  $m_i$  (equivalent) adults. Each adult possesses the living standard/equivalent income  $X_i/m_i$ . In other words, in this case we can represent the heterogeneous income distribution by a weighted distribution of equivalent income. We



always observe this (the same) outcome in the literature available on the meaning and characterization of the between-type progressive transfer principle.<sup>7</sup> Using this equivalence we actually transform the multidimensional distribution  $(\mathbf{X}, \mathbf{m})$  into an appropriate one-dimensional distribution  $(\bar{\mathbf{X}}, \mathbf{m})$ .

In general – if there are any returns to household size –  $m_i$  is strictly less than the number of individuals belonging to the household. Thus the weight attached to a single individual depends on the type of household he or she is living in. Individuals are not necessarily treated equally. Whenever we use relative equivalence scales we therefore get a conflict between the BTPT principle and the principle of individualism. The latter principle postulates that the individuals form the basis of the model. All individuals have to count equally, i.e., according to this principle in a household framework the weights have to be equal (proportional) to the number of individuals belonging to the household (cf. also Ebert (1997a), Shorrocks (2004), Trannoy (2003)). But our analysis demonstrates that weighting by the number of individuals contradicts the BTPT principle. Thus for a heterogeneous population, in which living standard is measured by means of (relative) equivalence scales, one has to make a choice between the BTPT principle and the principle of individualism. Since we are interested in the topic of redistribution we decide for the principle BTPT and confine ourselves to the social welfare functions (5).

In the following we can discard differentiability and consider all continuous welfare functions having this particular form. We therefore define

$$\mathcal{W}(\mathbf{m}) = \left\{ \sum m_i U(X_i/m_i) \mid U \text{ continuous, strictly concave, and strictly increasing} \right\}. \quad (6)$$

A social welfare function having the form (5) is the corresponding analogue to a welfare function of type (1). But the following analysis demonstrates that a welfare function  $W \in \mathcal{W}(\mathbf{m})$  also satisfies a more general concept of redistribution. We introduce:

**Definition 8**

Let  $(\mathbf{X}, \mathbf{m}) \in D^n \times \{\mathbf{m}\}$ .  $(\mathbf{Y}, \mathbf{m}) \in D^n \times \{\mathbf{m}\}$  is obtained from  $(\mathbf{X}, \mathbf{m})$  by a progressive redistribution scheme if there are nonnegative constants  $\lambda_{11}, \lambda_{12}, \dots, \lambda_{nn}$  such that

$$\frac{Y_i}{m_i} = \sum_{j=1}^n \lambda_{ij} \frac{X_j}{m_j} \text{ for } i=1, \dots, n, \quad \sum_{j=1}^n \lambda_{ij} = 1 \text{ for } i=1, \dots, n, \text{ and } \sum_{j=1}^n Y_j = \sum_{j=1}^n X_j.$$

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<sup>7</sup> Cf. also Hammond (1978) and Pyatt (1985, 1990) on this point.

By definition a progressive redistribution scheme redistributes income in such a way that the new *equivalent* incomes are a weighted sum of the old ones. It is clear that only household income can be redistributed. But, of course, this can be done under some restrictions on the equivalent incomes. It turns out that every between-type progressive transfer is a (simple) redistribution scheme (but not the other way around): Suppose that  $X_i/m_i < X_j/m_j$  for  $\mathbf{X} \in D^n \times \{\mathbf{m}\}$  and  $Y_i/m_i = (X_i + \varepsilon)/m_i \leq (X_j - \varepsilon)/m_j = Y_j/m_j$  for  $\varepsilon > 0$  and consider a transfer between household  $j$  and household  $i$ .

Then

$$Y_i/m_i = (1 - \lambda) X_i/m_i + \lambda X_j/m_j$$

and

$$Y_j/m_j = \kappa X_i/m_i + (1 - \kappa) X_j/m_j$$

for  $\lambda = \varepsilon / (m_i (X_j/m_j - X_i/m_i))$  and  $\kappa = m_i \lambda / m_j$ . Thus the transfer can be interpreted as a progressive redistribution scheme. Example 2 in Ebert (1999) demonstrates that the converse is not true. It is not always possible to generate a progressive redistribution scheme by a finite sequence of (between-type) progressive transfers.

For future use we formulate the corresponding principle

### **Principle of progressive redistribution schemes PRS( $\mathbf{m}$ )**

*Assume that  $(\mathbf{Y}, \mathbf{m}) \in D^n \times \{\mathbf{m}\}$  is generated from  $(\mathbf{X}, \mathbf{m}) \in D^n \times \{\mathbf{m}\}$  by a progressive redistribution scheme. Then  $W(\mathbf{Y}, \mathbf{m}) > W(\mathbf{X}, \mathbf{m})$ .*

The reader might foresee that progressive redistribution schemes are the proper tools to be used in the following since the welfare functions introduced above satisfy PRS( $\mathbf{m}$ ). We are able to establish

### **Theorem 4**

*Let  $m(a_1), \dots, m(a_\kappa)$  be given. The social welfare function  $W(\mathbf{X}, \mathbf{m}) = \Sigma V(X_i, m_i)$  satisfies PRS( $\mathbf{m}$ ) if and only if there is a strictly concave and increasing utility-of-income function  $U(X)$  such that  $V(X, m) = mU(X/m)$ .*

For a proof see the Appendix. If we impose PRS( $\mathbf{m}$ ) we get the same family of social welfare functions as before. On the other hand, a social welfare function having the form (5), in which

$U$  is strictly concave and increasing, does not only react positively to a between-type progressive transfer, but it also increases when any progressive redistribution scheme – which is a more general concept – is applied. Therefore we will now consider progressive redistribution schemes.

Having clarified these connections we next turn to a discussion of the redistribution of income in the framework presented.

## 4.2 Characterizations

In the following we assume that the population is fixed, i.e., the vector of attributes  $\bar{a} = (\bar{a}_1, \dots, \bar{a}_n)$  and therefore the vector of equivalence scales  $\mathbf{m} = (m_1, \dots, m_n)$  is given. We will now demonstrate that a progressive redistribution scheme is the proper generalization of a simple Pigou-Dalton progressive transfer to the heterogeneous framework. At first we introduce the concept of an  $\mathbf{m}$ -stochastic matrix.

### Definition 9

*A matrix  $B = (b_{ij}) \in \mathbb{R}_{++}^{n \times n}$  is called  $\mathbf{m}$ -stochastic if and only if  $\sum_i b_{ij} = 1$  and  $\sum_j b_{ij} m_j = m_i$  for all  $j \in N$  and, respectively,  $i \in N$ .*

It is obviously a generalization of the concept of a bistochastic matrix. But in this case the equivalence scales are taken into account by the column sums of the matrix. Again total income is not changed by the multiplication by an  $\mathbf{m}$ -stochastic matrix. If all households consist of only one adult (i.e.,  $m_1 = \dots = m_n = 1$ ) an  $\mathbf{m}$ -stochastic matrix is bistochastic.

We obtain

### HET 1

*Assume that  $(\mathbf{X}, \mathbf{m}), (\mathbf{Y}, \mathbf{m}) \in D^n \times \{\mathbf{m}\}$ . Then  $(\mathbf{Y}, \mathbf{m})$  is obtained from  $(\mathbf{X}, \mathbf{m})$  by means of a progressive distribution scheme if and only if there is an  $\mathbf{m}$ -stochastic  $B \in \mathbb{R}_{++}^{n \times n}$  such that  $\mathbf{Y} = B \cdot \mathbf{X}$ .*

Thus a progressive redistribution scheme can be represented by a corresponding  $\mathbf{m}$ -stochastic matrix (and conversely). The equivalence scales describing the attributes of the corresponding household types are an integral part of both concepts.

In the next step we introduce an appropriate definition of Lorenz dominance. A given heterogeneous income distribution can be described equivalently by the weighted distribution of

equivalent income  $(\bar{X}, \mathbf{m})$ . For this distribution we define a Lorenz curve in a natural way by  $LC(i/n, \mathbf{X}, \mathbf{m}) := \sum_{j=1}^i (m_{(j)}/\Sigma m_k) \bar{X}_{(j)}/\mu(\mathbf{X}, \mathbf{m})$  for  $i=0, \dots, n$  where  $(\bar{X}_{(j)}, \mathbf{m}_{(j)})$  is the heterogeneous distribution in which households, equivalent incomes, and equivalent scales are rearranged such that  $\bar{X}_{(i)} \leq \bar{X}_{(i+1)}$  for  $i=1, \dots, n-1$  and  $\mu(\mathbf{X}, \mathbf{m})$  denotes the weighted average equivalent income,

$$\mu(\mathbf{X}, \mathbf{m}) = \Sigma_i m_i (X_i/m_i) / \Sigma_k m_k.$$

Again the curve is interpolated linearly between  $i/n$  and  $(i+1)/n$ . This Lorenz curve can also be interpreted as a Lorenz curve for the distribution of equivalent income of the population of equivalent adults. Using this concept we introduce

**Definition 10**

$(\mathbf{Y}, \mathbf{m}) \in D^n \times \{\mathbf{m}\}$  (strictly) Lorenz dominates,  $(\mathbf{X}, \mathbf{m}) \in D^n \times \{\mathbf{m}\}$  i.e.,  $(\mathbf{X}, \mathbf{m}) \geq_{LD} (\mathbf{Y}, \mathbf{m})$  ( $(\mathbf{X}, \mathbf{m}) >_{LD} (\mathbf{Y}, \mathbf{m})$ ) if and only if  $LC(i/n, \mathbf{Y}, \mathbf{m}) \geq LC(i/n, \mathbf{X}, \mathbf{m})$  for  $i=0, \dots, n$  (and at least one inequality is strict).

This is obviously the feasible notion of dominance in the framework considered. Now things fit together. We get

**HET 2**

Assume that  $(\mathbf{X}, \mathbf{m}), (\mathbf{Y}, \mathbf{m}) \in D^n \times \{\mathbf{m}\}$  and  $\mu(\mathbf{X}, \mathbf{m}) = \mu(\mathbf{Y}, \mathbf{m})$ . Then  $(\mathbf{Y}, \mathbf{m})$  is obtained from  $(\mathbf{X}, \mathbf{m})$  by a progressive redistribution scheme if and only if  $(\mathbf{Y}, \mathbf{m})$  strictly Lorenz dominates  $(\mathbf{X}, \mathbf{m})$ .

This result is the analogue to HOM 2. It shows that in the multidimensional framework between-type progressive transfers are also important (since they are specific progressive redistribution schemes), but that they are not sufficiently general in order to explain the generation of any Lorenz dominating income distribution  $(\mathbf{Y}, \mathbf{m})$  given a distribution  $(\mathbf{X}, \mathbf{m})$ . Furthermore, it should not come as a surprise that in this case the average *equivalent* incomes of both distributions have to be identical.

Moreover we now also obtain a normative description of a redistribution scheme and of Lorenz dominance by

### HET 3

Assume that  $(X, \mathbf{m}), (Y, \mathbf{m}) \in D^n \times \{\mathbf{m}\}$  and  $\mu(X, \mathbf{m}) = \mu(Y, \mathbf{m})$ . Then  $(Y, \mathbf{m})$  is generated from  $(X, \mathbf{m})$  by means of a progressive redistribution scheme if and only if  $W(Y, \mathbf{m}) > W(X, \mathbf{m})$  for all  $W \in \mathcal{W}(\mathbf{m})$ .

and, respectively,

### HET 4

Assume that  $(X, \mathbf{m}), (Y, \mathbf{m}) \in D^n \times \{\mathbf{m}\}$  and  $\mu(X, \mathbf{m}) = \mu(Y, \mathbf{m})$ . Then  $(Y, \mathbf{m}) >_{LD} (X, \mathbf{m})$  if and only if  $W(Y, \mathbf{m}) > W(X, \mathbf{m})$  for all  $W \in \mathcal{W}(\mathbf{m})$ .

Thus again the results prove that the approach of generalizing the concepts to the heterogeneous framework by considering the *weighted distribution of equivalent income* is feasible and makes sense.

This statement is finally supported by reconsidering generalized Lorenz dominance. We define the generalized Lorenz curve by  $GL(i/n, X, \mathbf{m}) := \sum_{j=1}^i (m_{(j)} / \Sigma m_k) \bar{X}_{(j)}$  for  $i = 0, \dots, n$  and generalized Lorenz dominance by

### Definition 11

$(Y, \mathbf{m}) \in D^n \times \{\mathbf{m}\}$  (strictly) generalized Lorenz dominates  $(X, \mathbf{m}) \in D^n \times \{\mathbf{m}\}$ , i.e.,  $(Y, \mathbf{m}) \geq_{GL} (X, \mathbf{m})$  ( $(Y, \mathbf{m}) >_{GL} (X, \mathbf{m})$ ) if and only if  $GL(i/n, Y, \mathbf{m}) \geq GL(i/n, X, \mathbf{m})$  for  $i = 0, \dots, n$  (and at least one inequality is strict).

Then we also obtain an analogue to HOM 5:

### HET 5

Assume that  $(X, \mathbf{m}), (Y, \mathbf{m}) \in D^n \times \{\mathbf{m}\}$ . Then  $(Y, \mathbf{m}) >_{GL} (X, \mathbf{m})$  if and only if  $W(Y, \mathbf{m}) > W(X, \mathbf{m})$  for all  $W \in \mathcal{W}(\mathbf{m})$ .

Here, of course, the average equivalent incomes may differ. A generalized Lorenz curve always reflects the distribution of the (absolute) equivalent incomes (inherent inequality). Again distributional and efficiency aspects are combined in one criterion. The generalized Lorenz curve reacts to changes in the size and the distribution of living standard.

Finally we turn to taxation. For a heterogeneous population things are, of course, more complicated than above. Different household types have to be treated in a just manner. For horizontal equity we have to identify those being equal. We describe a household's needs by means of the living standard attained. Let  $T : D \times \{m(a_1), \dots, m(a_k)\} \rightarrow \mathbb{R}$  be an income tax schedule.  $T(X, m)$  is equal to the tax liability of a household of type  $m$  (the type is reflected by the equivalence scale) and household income  $X$ . Then we postulate

### Horizontal equity

Assume that two households  $i$  and  $j$  possess the same living standard, i.e.,  $X_i/m_i = X_j/m_j$  for any  $X_i, X_j \in D$  and  $m_i, m_j \in \{m(a_1), \dots, m(a_k)\}$ . Then  $T$  is horizontally equal if

$$\frac{X_i - T(X_i, m_i)}{m_i} = \frac{X_j - T(X_j, m_j)}{m_j}.$$

Thus households which reach the same level of living standard have to be treated equally, i.e., they also have to attain the same living standard after taxation. The implications of this principle can be derived directly. Suppose that  $t(X)$  denotes the tax schedules for single adults (the reference type having  $m(a_r) = 1$ ). Then horizontal equity and  $X_i/m_i = X_i/1 = X_j/m_j$  imply that

$$\frac{X_i - T(X_i, m_i)}{m_i} = X_i - t(X_i) = \frac{X_j - T(X_j, m_j)}{m_j} \quad (7)$$

and therefore  $T(X, m) = mt(X/m)$ . The tax liabilities of different types have to be interdependent. The liability of a type  $m$  household with income  $X$  has to be equal to the  $m$  fold liability of a reference household with equivalent income  $X/m$ . In this case the tax burden per equivalent adult is always the same.<sup>8</sup> Thus a horizontally equal tax schedule is given by  $t(X)$  for single adults and  $mt(X/m)$  for households of type  $m$  and we get 'family splitting'. Accordingly we define

### Definition 12

A tax schedule  $T(X, m) = mt(X/m)$  is progressive if and only if the residual elasticity of  $t(X)$  is less than or equal to unity for all  $X \in D$ .

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<sup>8</sup> See Ebert and Lambert (2004) for a deeper discussion of horizontal equity in heterogeneous populations.

In this framework we obtain

## HET 6

*Let a tax schedule  $T(X, m)$  be horizontally equal. Then  $T(X, m)$  is progressive if and only if  $(X - T(X, m), m) \geq_{LD} (X, m)$  for all income distributions  $(X, m) \in D^n \times \{m\}$ .*

A progressive tax schedule is at the same time inequality reducing (and conversely). But in this case we have to emphasize that the principle of horizontal equity plays an important role. It implies and requires that the tax schedules for different household types are properly defined. This condition has to be met since otherwise between-type redistributions are not necessarily inequality reducing.

The relationship between progressive taxation and inequality reduction in a heterogeneous framework can also be illuminated differently. In a model, which is a little bit more restricted than ours, Ebert and Moyes (2000) prove that income taxation is inequality reducing if and only if the tax schedule possesses the form  $T(X, a) = (E^r)^{-1}(t(E^r(X, a)), a_r)$  where  $t(X)$  is a progressive tax schedule for the reference type. If in addition inequality reduction is to be independent of the choice of the reference type, living standard has to be measured by means of equivalence scales and we get the form of the tax schedule  $T$  introduced above:  $T(X, m) = mt(X/m)$  (their Proposition 4.2).

## 5. Conclusion

The analysis has demonstrated that the redistribution of income in a heterogeneous population is formally similar to that in a homogeneous one whenever equivalent income is considered. There are slight differences if (relative) equivalence scales  $m$  are employed. In particular one has to attach the number of equivalent individuals to a household's equivalent income. This proceeding guarantees that the corresponding between-type progressive transfer principle and the principle PRS( $m$ ) are satisfied. There are also several empirical papers following this route (e.g. Slesnick (1994) and Szulc (1995)).

The above investigation has concentrated on social welfare. We obtain analogous results when we consider inequality (Ebert (1995, 2004)) or poverty (cf. Ebert (2004, 2005, 2010b)). Here it does not play a role whether the inequality (or poverty) measure is defined directly (or indirectly via a social welfare function). A principle of BTPT always requires that the heterogeneous income distribution is transformed into a distribution of equivalent income for a

population of equivalent individuals. The situation is slightly different if other forms of transfer between households of different types are considered (Ebert (2000b, 2007)).

Finally it should be mentioned that in the literature some results have also been derived for absolute equivalence scales. These scales are not as popular as relative scales, but the results for absolute scales are in principle analogous to those proven for relative ones (cf. Ebert (1997b, 2004, 2007), Ebert and Moyes (2000, 2002, 2003)).

### Appendix: Proof of Theorem 4

(i) Let  $\Sigma_i V(X_i, m_i)$  satisfy PRS( $m$ ). Then it also satisfies BTPT( $m$ ). Theorem 3 implies that  $V(X, m) = mU(X/m)$ .

(ii) Conversely, suppose that  $(Y, m)$  is generated from  $(X, m)$  by means of a redistribution scheme. HET 1 implies that there is an  $m$ -stochastic matrix s.t.  $Y = B \cdot X$ .

Then  $Y_i/m_i = \Sigma_j b_{ij} X_j/m_j = \Sigma_j (b_{ij} m_j/m_i) X_j/m_j$ .

Concavity of  $U$ ,  $\Sigma_j b_{ij} m_j/m_i = 1$ , and  $\Sigma_i b_{ij} = 1$  imply:

$$\begin{aligned} W(Y, m) &= \Sigma_i m_i U(Y_i/m_i) = \Sigma_i m_i U\left(\Sigma_j (b_{ij} m_j/m_i) X_j/m_j\right) \\ &> \Sigma_i m_i \Sigma_j (b_{ij} m_j/m_i) U(X_j/m_j) = \Sigma_j (\Sigma_i b_{ij}) m_j U(X_j/m_j) = W(X, m) \end{aligned}$$

□

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